Nonlinear waves in granular chains

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GDR DYNOLIN, 17 October 2013, Lille

Outline :

I - Localized waves in granular crystals : solitary waves, breathers

II - Breather dynamics, discrete p-Schrödinger equation

I - Localized waves in granular crystals

= granular media with spatial order

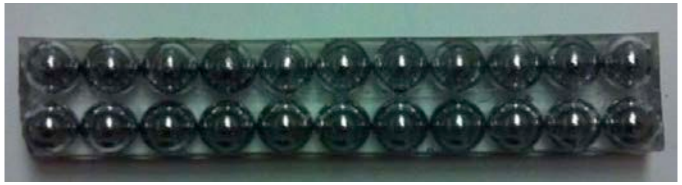
Experiments on stress wave propagation :

C. Daraio's group (Caltech, ETH)

Experiments at UIUC : beads in elastic matrix

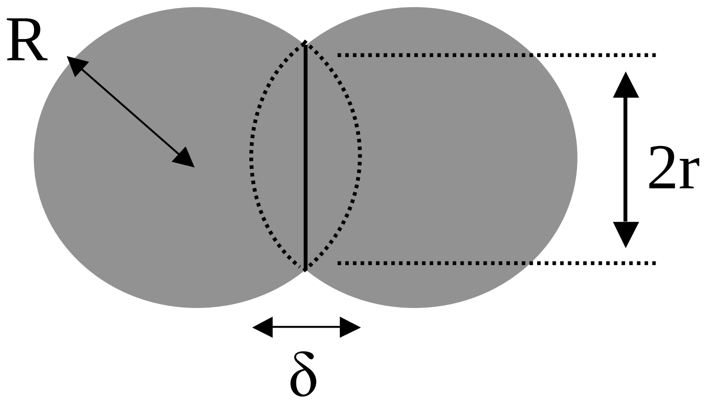
⇨ local restoring force, coupled chains

Groups of W. Kriven, A. Vakakis, J. Lambros



Applications : tunable media ⇨control of nonlinear stress waves

Source of nonlinearity : Hertz contact force



Hooke's law : Hertz force $\approx r\,\delta$

Geometric nonlinearity : $r \approx \sqrt{R\delta}$

$\Rightarrow$ Hertz force $\approx \delta^{3/2}$

- Fully nonlinear
- Unilateral
- Not C^2 at $\delta=0$

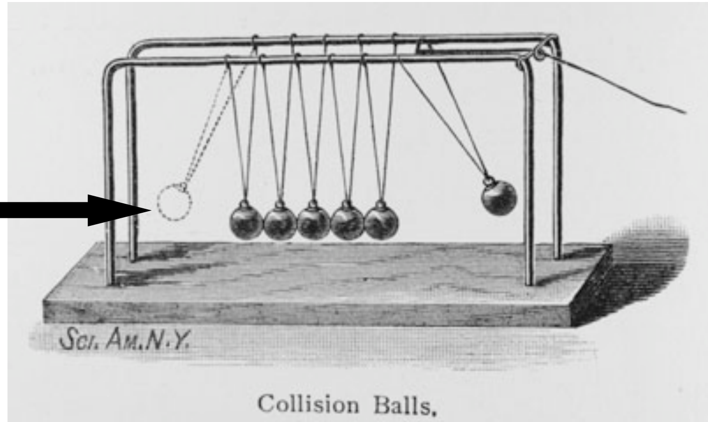
⇨ Nonlinear models for chains of beads delicate to analyze:
chains of point masses coupled by Hertzian potentials

⇨ Localized waves can be generated in simulations and experiments:
solitary waves, breathers,…

Solitary wave in Newton's cradle

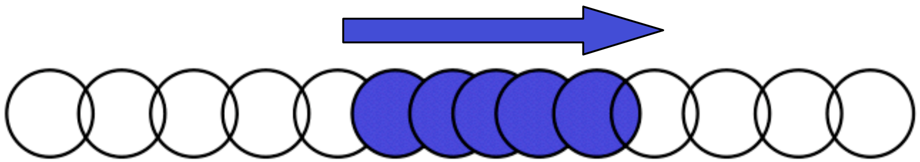
Newton's cradle :

Impacting bead



bead ejection

Nesterenko (84) :
propagation of a solitary wave



Definition :

contact forces = $f(n - c\,t)$, $\lim_{|\xi|\to\infty} f(\xi) = 0$

(n: bead index)

First experimental studies of Nesterenko's solitary wave :
Lazaridi and Nesterenko (85), Coste, Falcon and Fauve (97), Falcon et al (98)

An application : granular shock absorber

Experiments with tapered chains :

impact ➔

Melo et al,
Phys Rev E 73,
041305 (2006)

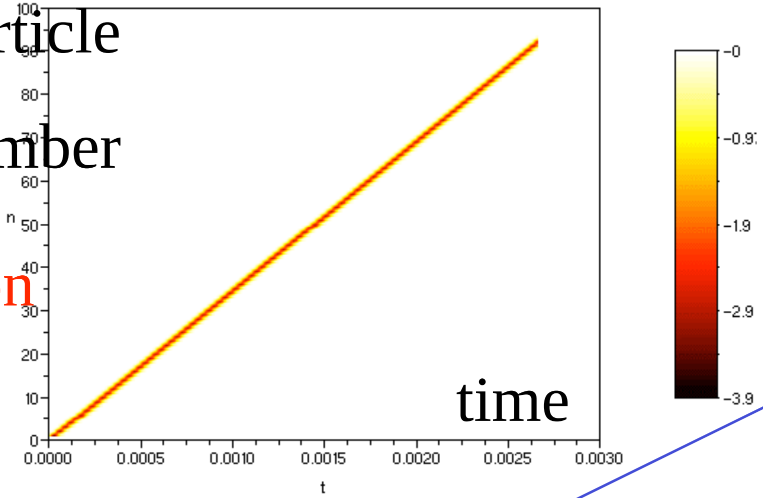
Numerical computation of contact forces :

homogeneous chain

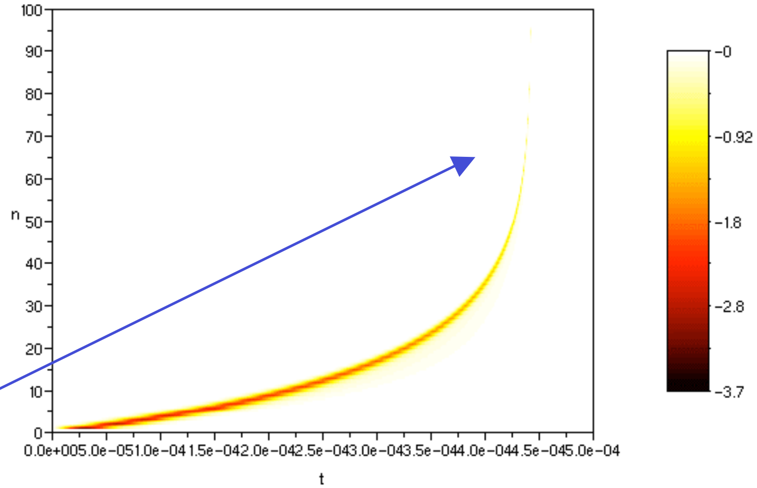
Particle number

compression pulse ➔

time



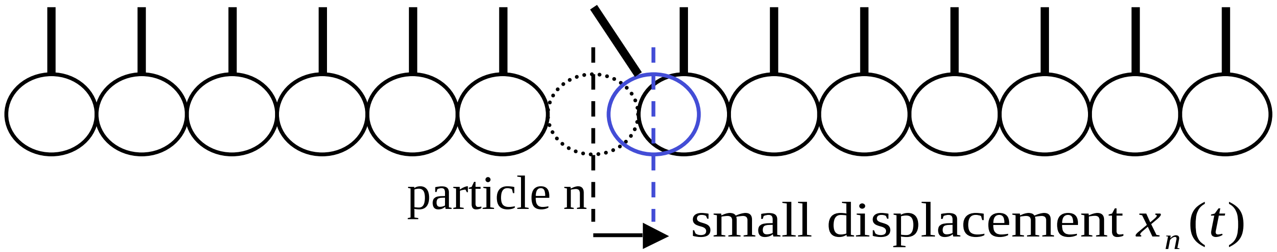
exponentially tapered chain



pulse attenuation during propagation (shock absorber)

Model : Fermi-Pasta-Ulam (FPU) lattice with Hertzian potential

Newton's cradle under gravity :



$$m\,\ddot{x}_n + k\,x_n = \gamma(x_{n-1} - x_n)_+^{3/2} - \gamma(x_n - x_{n+1})_+^{3/2} \qquad (a)_+ = \mathrm{Max}(a,0)$$

Separation of time scales : local oscillations (s), binary collisions (0.1 ms)

⇨ local potential neglected during primary pulse propagation

$$m\,\ddot{x}_n = \gamma(x_{n-1} - x_n)_+^{3/2} - \gamma(x_n - x_{n+1})_+^{3/2}$$

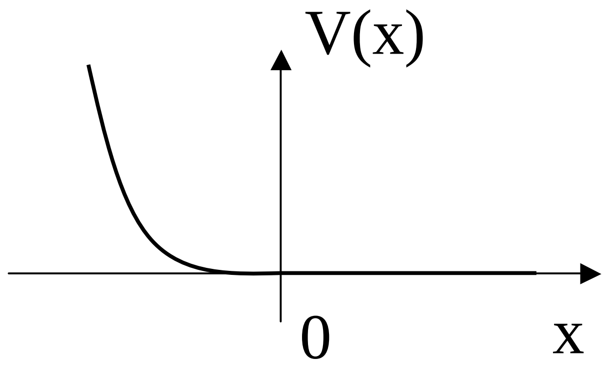
(Fermi-Pasta-Ulam - Hertz)

Solitary waves in FPU lattice with Hertzian potential

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \qquad n \in \mathbb{Z}$$

$$V(x) = \frac{1}{1+\alpha}(-x)_+^{1+\alpha}$$

$\alpha > 1$ (classical case : $\alpha = 3/2$)



$V''(0) = 0 \Rightarrow$ no sine waves in linearized system ("sonic vacuum")

Existence theorems for solitary waves :

Friesecke-Wattis '94, MacKay '99, Stefanov-Kevrekidis '12

Purely nonlinear Hertz force ⇨ unusual properties of solitary waves

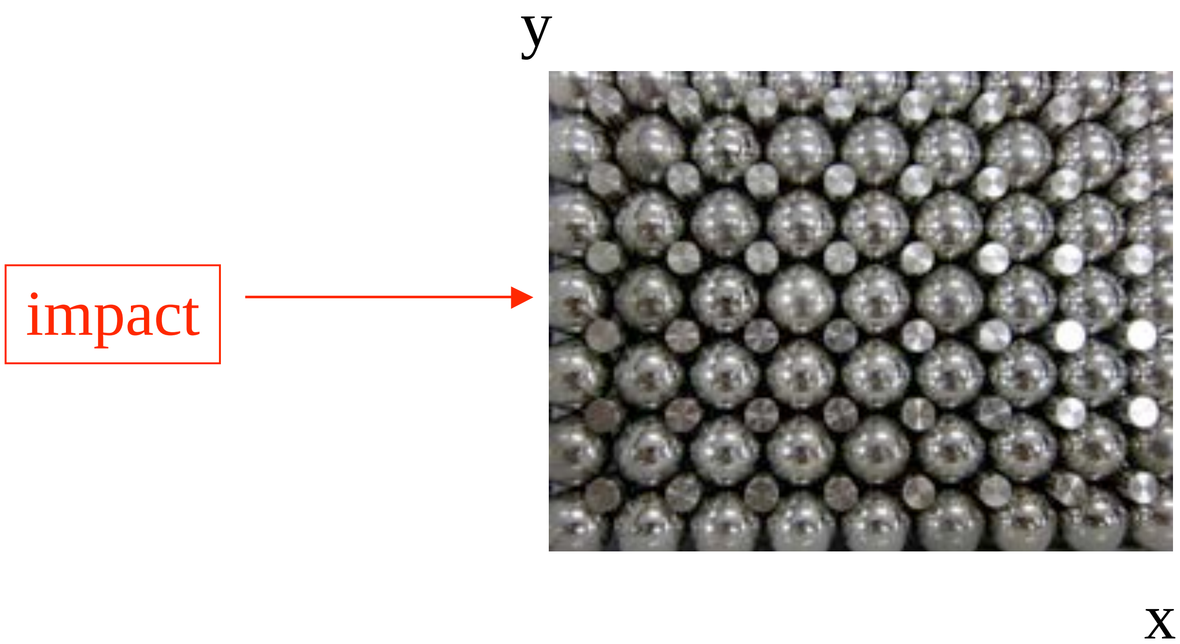
width independent of amplitude, double exponential spatial decay, …

English-Pego '05, Stefanov-Kevrekidis '12

Shock propagation in higher dimensional granular systems :

C. Daraio’s group (Caltech, ETH) :

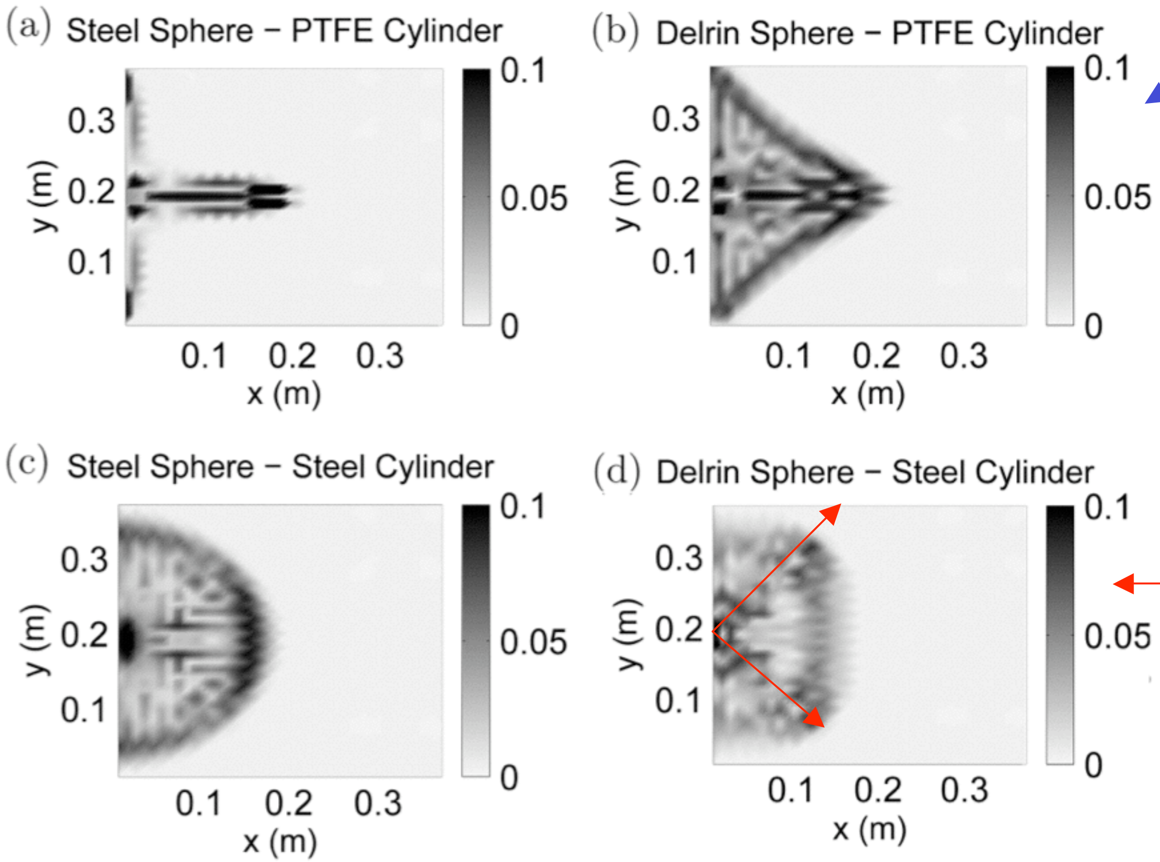
Experiments on 2D packing of spheres with interstitial cylinders



Shock propagation in higher dimensional granular systems :

Different characteristics of stress wave fronts

depending on materials :



Numerical simulations

(particle velocities)

Leonard and Daraio,
PRL 108, 214301 (2012)

Significant energy

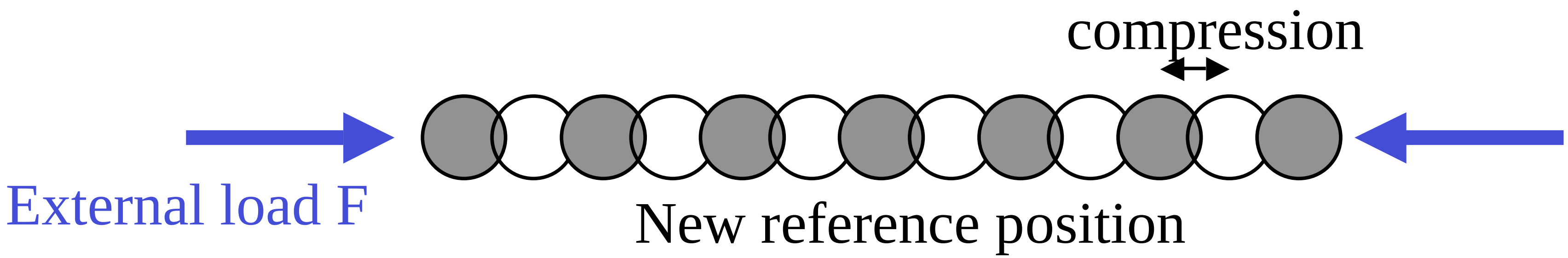
deviation

towards the edges

Breathers in granular chains under precompression

Breathers in diatomic chains - Experiments : Boechler et al ‘10,

Simulations : Theocharis et al, Phys. Rev. E 82, 2010

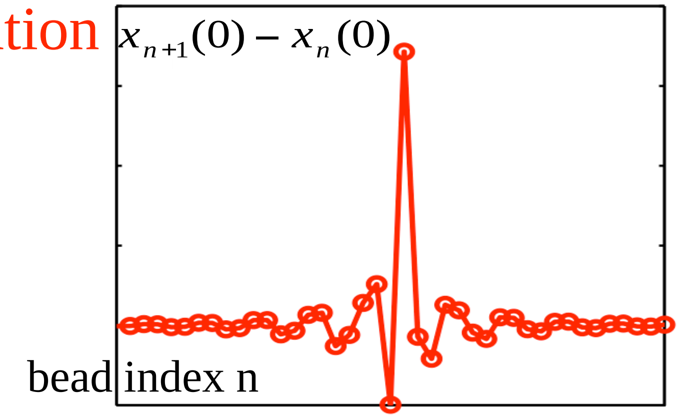


Breather : time-periodic spatially localized oscillation

displacements from reference position :

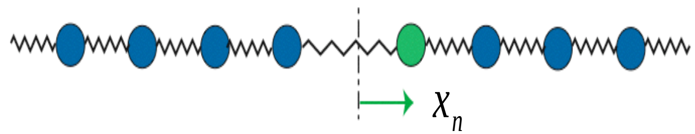
$$x_n(t+T) = x_n(t)$$

$$\lim_{n \to \pm\infty} x_{n+1}(t) - x_n(t) = 0$$



precompression $\Rightarrow$ classical Fermi - Pasta - Ulam

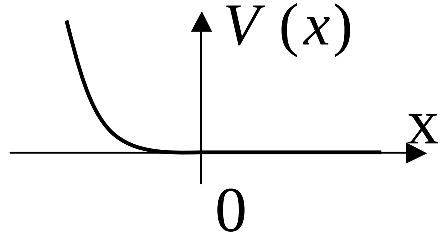
model : force $= k_F(x_{n+1} - x_n) + O(x_{n+1} - x_n)^2$



Without precompression : no breathers

$$m_n \ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \quad n \in \mathbb{Z}$$

Hertz potential : unilateral



No nontrivial T-periodic breather solutions satisfying $\lim_{n \to \pm\infty} \|x_n - x_{n-1}\|_{L^\infty(0,T)} = 0$

J., Kevrekidis, Cuevas,

Physica D 251 (2013)

Quick proof :

average interaction force : $f_n = \frac{1}{T}\int_0^T V'(x_n - x_{n-1})\, dt \to 0$ as $n \to \infty$ (localization)

f_n independent of $n \quad \Rightarrow \quad f_n = 0$

$V' \leq 0 \quad \Rightarrow \quad V' = 0 \quad \Rightarrow$ beads in free flight $\Rightarrow \frac{\mathrm{d}x_n}{\mathrm{dt}} = 0 \qquad \square$

II - Breathers in Hertzian chains with confining potentials

Outline :

1- Numerical simulations

2- Analysis of energy localization :

asymptotic model for small amplitude solutions : discrete p-Schrödinger equation

existence of small amplitude breathers

Time-periodic breathers in Hertzian chains with local potentials

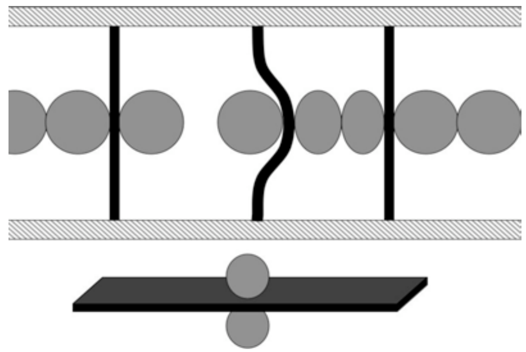
J., Kevrekidis, Cuevas ‘13

$$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

$$(a)_+ = \mathrm{Max}(a,0)$$

Stiff Newton’s cradle

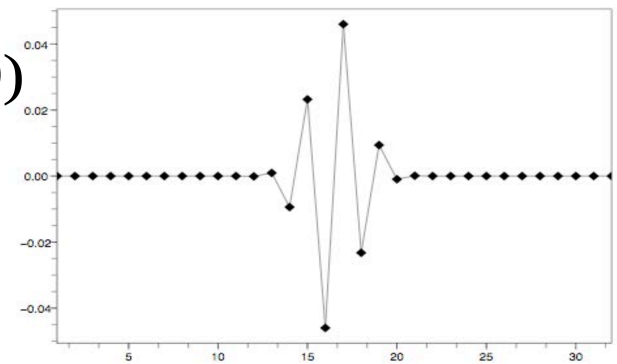
with cantilevers :



Breather computation by Newton’s method :

Bond-centered breather

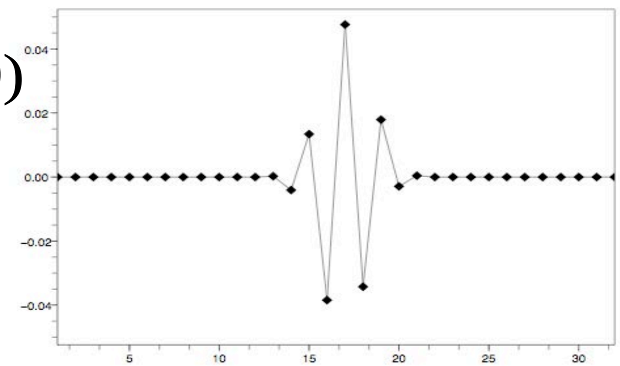
$x_n(0)$



n

Site-centered breather

$x_n(0)$



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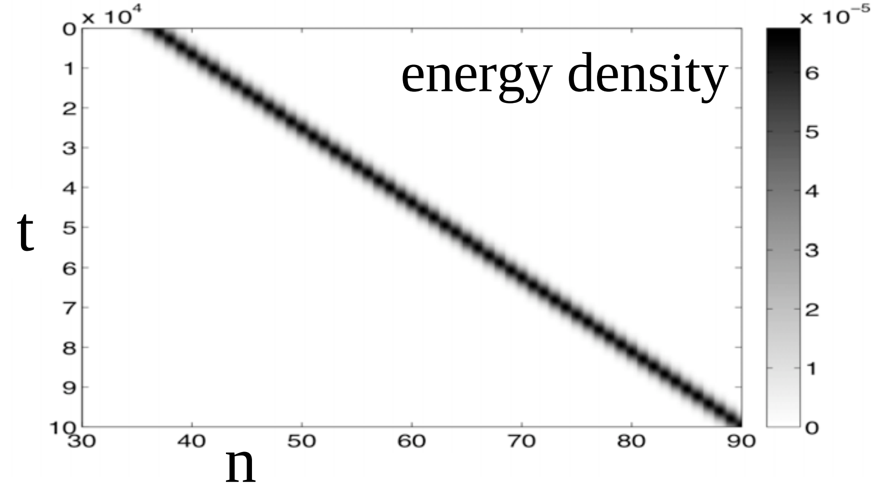
Unusual properties : double exponential decay, high mobility,…

Breather motion

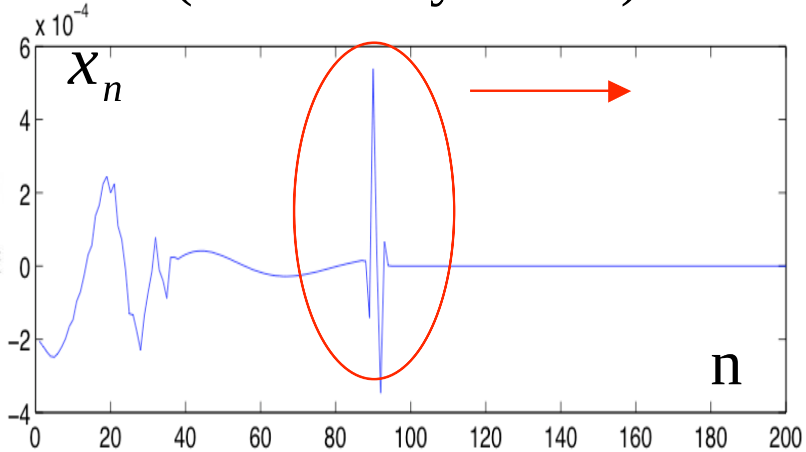
Small perturbation of a stable breather

(energy +0.01%) ⇨ traveling breather :

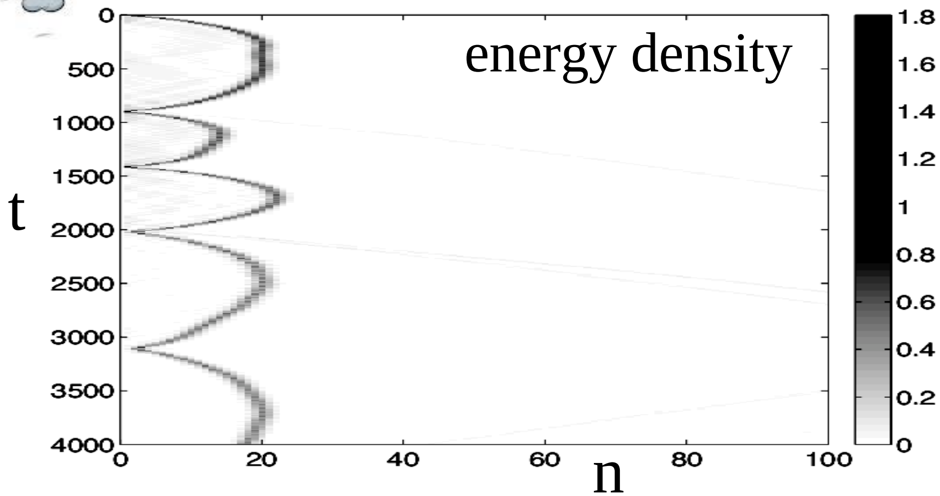


Moderate impacts generate traveling breathers

(not solitary waves)



Impact



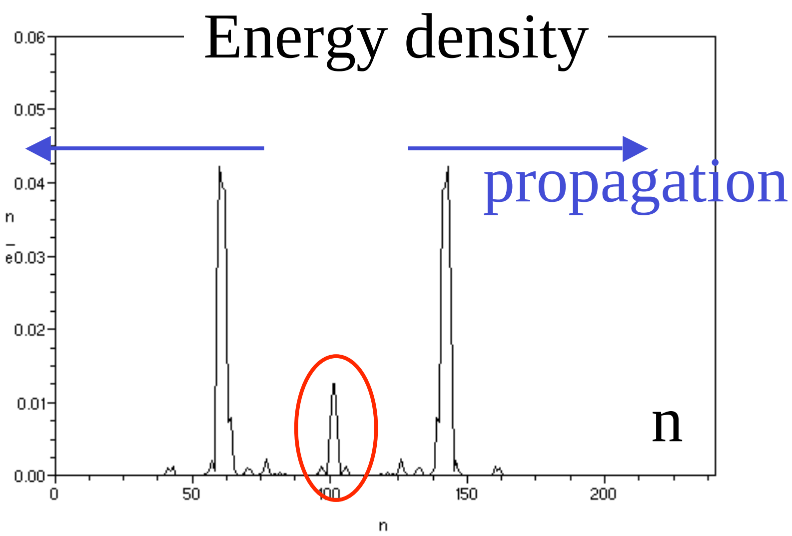
direction-reversing traveling breather :

hard local potential $\frac{x^2}{2}+\frac{x^4}{4}$

Static breather generation

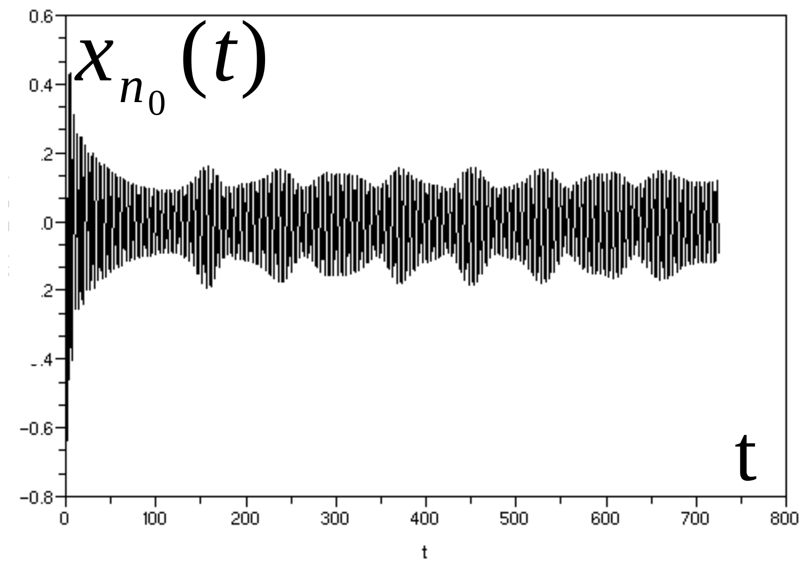
Initial compression of two beads

$(n = n_0 \text{ and } n = n_0 + 1)$



Breather
(non-propagating)

Initially compressed bead



Quasiperiodicity ?

Derivation of a simplified asymptotic model :

(N.C.) $$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{\alpha} - (x_n - x_{n+1})_+^{\alpha}$$ $(\alpha > 1)$

Leading order solutions (small amplitude ε) :

$$x_n^{A,\varepsilon}(t) = \varepsilon A_n(\varepsilon^{\alpha-1} t) e^{it} + \varepsilon \overline{A}_n(\varepsilon^{\alpha-1} t) e^{-it}$$ slow time : $\tau = \varepsilon^{\alpha-1} t$

Collect terms $O(\varepsilon^{\alpha}) \times e^{it}$ ⇨ **discrete p-Schrödinger (DpS) equation (J.'11)**

$$2\tau_0 \, i \partial_\tau A_n = (A_{n+1} - A_n) | A_{n+1} - A_n |^{\alpha-1} - (A_n - A_{n-1}) | A_n - A_{n-1} |^{\alpha-1}$$

Continuum limit : $i \partial_\tau A = \partial_\xi (\partial_\xi A | \partial_\xi A |^{p-2})$ with $p = \alpha + 1$ (p-Laplace operator)

What we have gained with DpS:

$\Rightarrow$ fast oscillations averaged,

$\Rightarrow$ unilateral character of interactions suppressed by averaging,

$\Rightarrow$ phase invariance, conservation of ℓ_2 norm, scale invariance

Newton's cradle vs discrete p-Schrödinger : error bounds

Infinite chain ($n \in \mathbb{Z}$),

phase space = sequence space ℓ_p with $p \in [1,\infty]$, $\|A\|_p = \left(\sum_{n=-\infty}^{+\infty} |A_n|^p \right)^{1/p}$

The DpS equation approximates true O(ε) solutions of N.C.

up to an error $O(\varepsilon^\alpha)$, over long times $O(\varepsilon^{1-\alpha})$:

Theorem (Bidégaray-Fesquet, Dumas, J. '13)

Fix a solution of DpS: $A_n(\tau): \; [0,T] \to \ell_p(\mathbb{Z})$

For all ε small enough, the solutions of N.C. with initial conditions

$(x_n(0),\dot{x}_n(0))_n = (x_n^{A,\varepsilon}(0),\dot{x}_n^{A,\varepsilon}(0))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ satisfy :

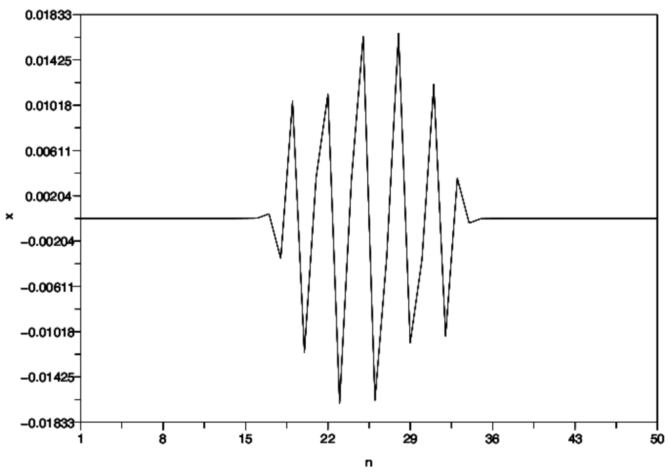
$(x_n(t),\dot{x}_n(t))_n = (x_n^{A,\varepsilon}(t),\dot{x}_n^{A,\varepsilon}(t))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ uniformly in $t \in [0, T\varepsilon^{1-\alpha}]$

Numerical comparison N.C. - DpS (α=3/2)

Initial displacements :

(zero initial velocities)

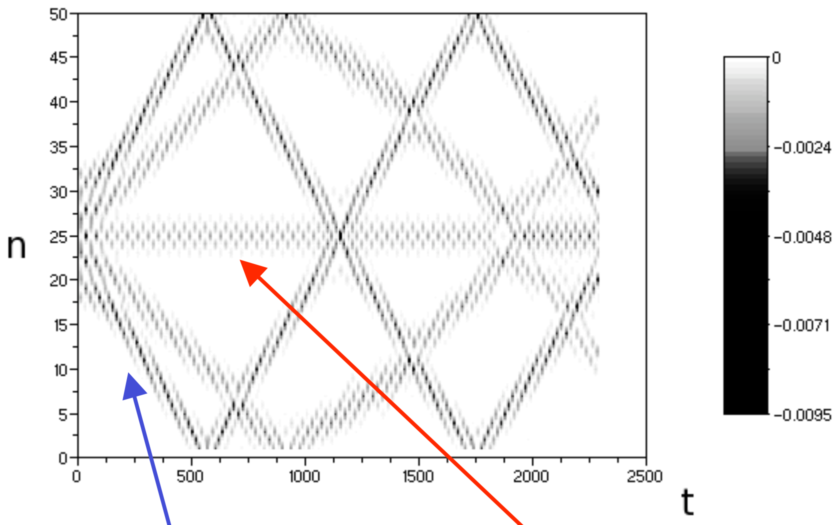
$x_n(0)$



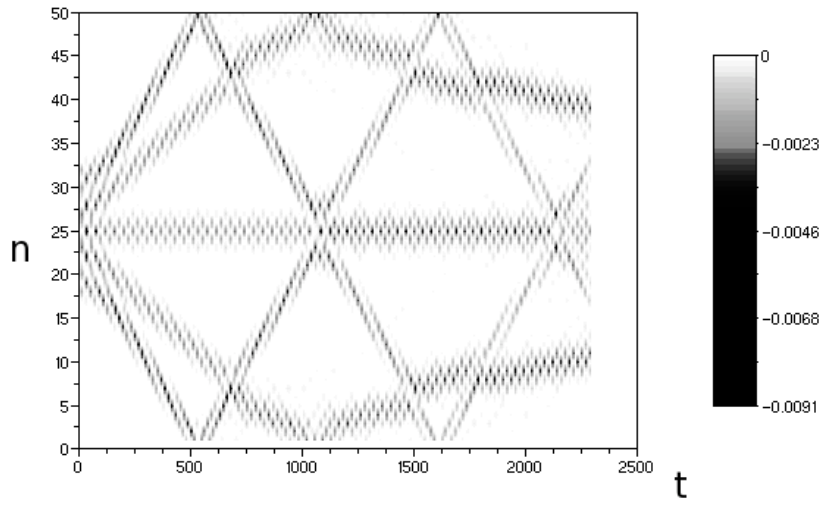
n

Interaction forces (grey levels) :

Newton's cradle :



DpS approximation:



traveling breather   breather (non-propagating)

Localized solutions of the DpS equation :
1 - Absence of scattering

DpS equation $(\alpha > 1)$:

$$i\,\partial_\tau A_n = (A_{n+1} - A_n)\,|A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1})\,|A_n - A_{n-1}|^{\alpha-1}$$

Theorem (Bidégaray-Fesquet, Dumas, J. '13) :

DpS is globally well-posed in $\ell_2(\mathbb{Z})$. If $A(0) \neq 0$, then

for all times : $$\|A(\tau)\|_\infty \geq \left(\frac{\left\| \frac{1}{2}\delta^+ A(0) \right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2} \right)^{\frac{1}{\alpha-1}}$$

$$(\delta^+ A)_n = A_{n+1} - A_n, \qquad \|\mathrm{A}\|_\mathrm{p} = \left(\sum_{\mathrm{n}=-\infty}^{+\infty} |A_n|^p \right)^{1/p}, \qquad \|\mathrm{A}\|_\infty = \operatorname*{Sup}_n |A_n|$$

Proof of absence of scattering for DpS equation :

Use conservation of energy $H = \sum_{n=-\infty}^{+\infty} |A_{n+1} - A_n|^{\alpha+1}$ and $\|A\|_2^2 = \sum_{n=-\infty}^{+\infty} |A_n|^2$

$$\|\delta^+ A(0)\|_{\alpha+1} = H^{\frac{1}{\alpha+1}} = \|\delta^+ A(\tau)\|_{\alpha+1} \qquad \text{(energy conservation)}$$

$$\leq 2\|A(\tau)\|_{\alpha+1} \qquad \text{(triangular inequality)}$$

$$\leq 2\left(\|A(\tau)\|_\infty\right)^{1-\frac{2}{1+\alpha}} \left(\|A(\tau)\|_2\right)^{\frac{2}{1+\alpha}} \qquad \text{(interpolation inequality)}$$

$$\leq 2\left(\|A(\tau)\|_\infty\right)^{1-\frac{2}{1+\alpha}} \left(\|A(0)\|_2\right)^{\frac{2}{1+\alpha}} \qquad \text{(conserved } \ell_2 \text{ norm)}$$

$$\Rightarrow \|A(\tau)\|_\infty \geq \left(\frac{1}{2}\|\delta^+ A(0)\|_{\alpha+1}\right)^{\frac{\alpha+1}{\alpha-1}} \left(\|A(0)\|_2\right)^{\frac{2}{1-\alpha}} = \left(\frac{\left\|\frac{1}{2}\delta^+ A(0)\right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2}\right)^{\frac{1}{\alpha-1}}$$

□

Localized solutions of the DpS equation :
2 - Breather solutions (time-periodic)

$$\text{DpS} : \quad i\partial_\tau A_n = (A_{n+1} - A_n)\,|A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1})\,|A_n - A_{n-1}|^{\alpha-1}$$

Time-periodic solutions to DpS : $A_n(\tau) = a_n e^{i\tau}$ $\quad \boxed{a_n \in \mathbb{R}}$

Stationary (real) DpS equation :

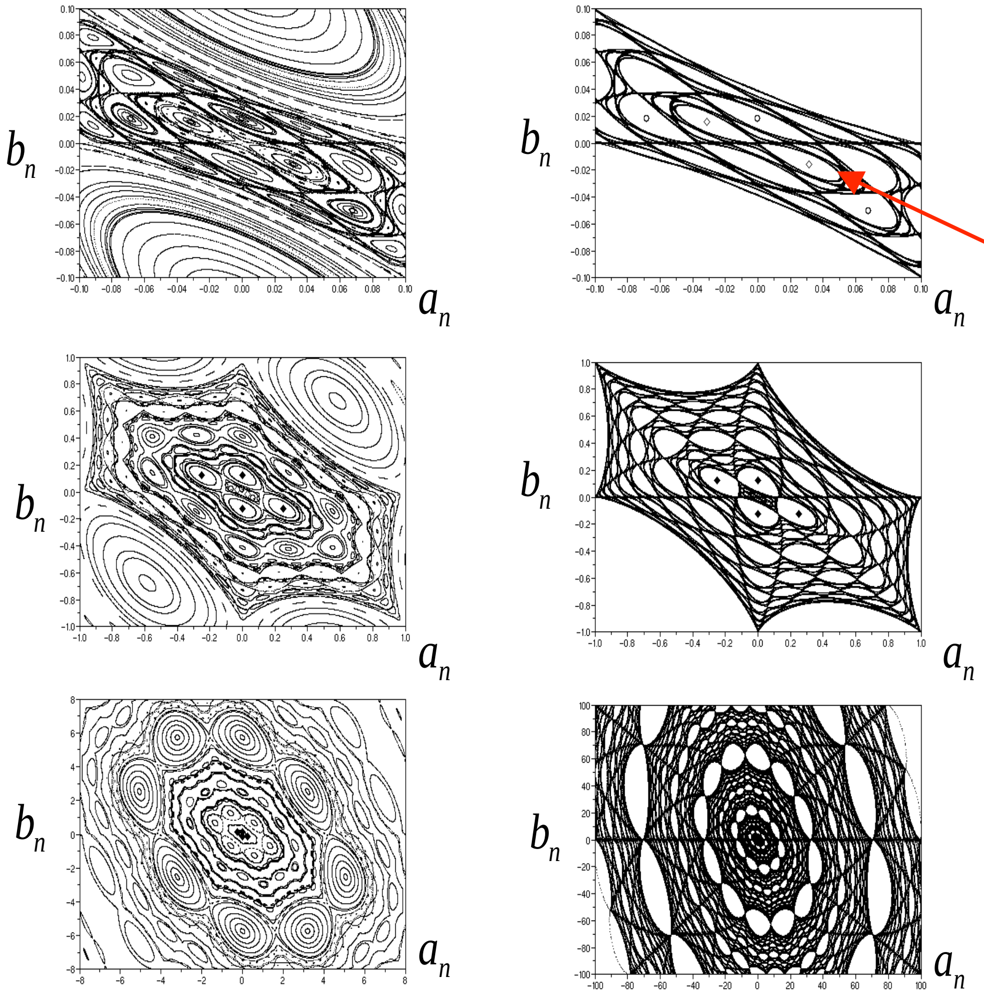
$$-a_n = \underbrace{(a_{n+1} - a_n)\,|a_{n+1} - a_n|^{\alpha-1}}_{b_n} - (a_n - a_{n-1})\,|a_n - a_{n-1}|^{\alpha-1}$$

Spatial dynamics : stationary real DpS ⇔ 2D mapping

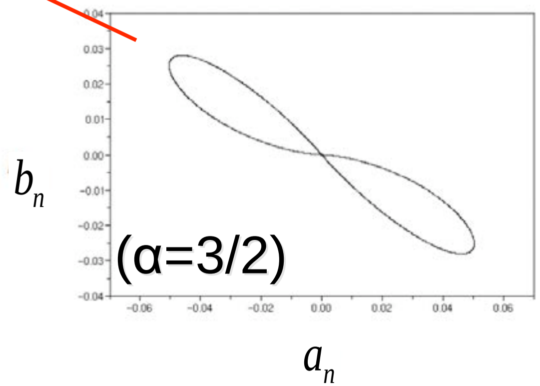
$(a_{n+1}, b_{n+1}) = G(a_n, b_n)$ $\qquad$ G reversible, area-preserving, not differentiable at the origin

Stationary Dps equation :
some orbits of the « spatial map » G

Zoom towards (0,0)



Stable and unstable manifolds of (0,0) intersect :



⇨orbits homoclinic to 0

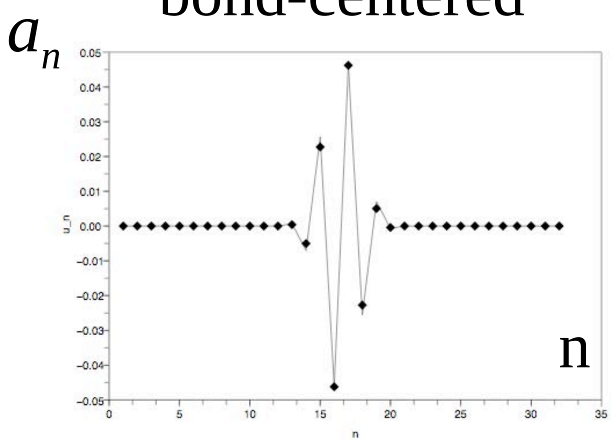
$$(a_n, b_n) \xrightarrow[n \to \pm\infty]{} 0$$

Theorem :

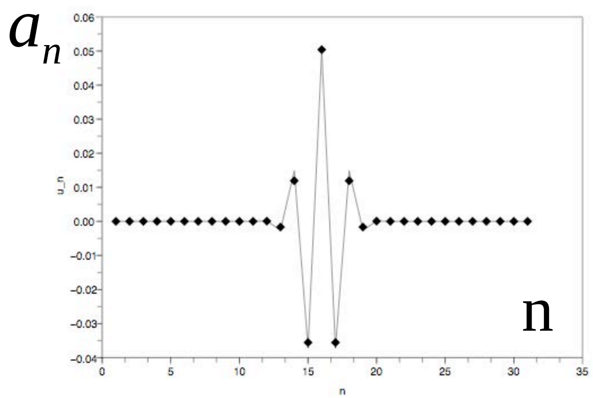
existence of reversible homoclinics

(J. and Starosvetsky ‘12)

bond-centered



site-centered



Method : $y_n = (-1)^n b_n$ solution of generalized stationary discrete NLS equation

$$y_{n+1} - 2y_n + y_{n-1} + U'(y_n) = 0 \Rightarrow$$ shooting method (Qin and Xiao '07)

$$U(y) = -\frac{\alpha}{\alpha+1} |y|^{1+1/\alpha} + 2y^2$$ nonsmooth double - well potential

Orbits homoclinic to 0
for stationary DpS equation

⇨ long-lived breather solutions of N.C. :

$$x_n(t) = 2\varepsilon a_n \cos[(1 + \frac{\varepsilon^{\alpha-1}}{2\tau_0}) t] + O(\varepsilon^\alpha)$$

over long times $t \approx \varepsilon^{1-\alpha}$

Conclusion

We have studied granular chains with local potentials
(beads attached to stiff cantilevers, cables, elastic matrix…)

❖model : nonlinear lattice, Hertzian interactions and local potential

❖strongly localized breathers (static/moving) easily generated in simulations

❖ new asymptotic model : discrete p-Schrödinger equation
-in Newton's cradle : describes small solutions over long times
-absence of scattering, existence of time-periodic breathers

Future directions :

incorporate dissipation and spatial inhomogeneities,

experimental study of breathers and applications

THANK YOU !

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J., Cuevas, Kevrekidis, in conf. proceedings Nonlinear Theory and its Applications (NOLTA 2012), arXiv:1301.1769

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Stefanov and Kevrekidis, J. Nonlinear Sci. 22 (2012), 327-349

J. and Pelinovsky, Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, 2013, arXiv:1307.3837

Further reading

Reviews and monographs on solitary waves and impacts in granular chains :

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E. Falcon, Comportements dynamiques associés au contact de Hertz : processus collectifs de collision et propagation d'ondes solitaires dans les milieux granulaires, PhD thesis, Université Claude Bernard Lyon 1 (1997)