

Nonlinear waves in granular chains

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GDR DYNOLIN, 17 October 2013, Lille

Outline :

I - Localized waves in granular crystals : solitary waves, breathers

II - Breather dynamics, discrete p-Schrödinger equation

I - Localized waves in granular crystals

= granular media with spatial order



Experiments on stress wave propagation :

C. Daraio's group (Caltech, ETH)

Experiments at UIUC : beads in elastic matrix

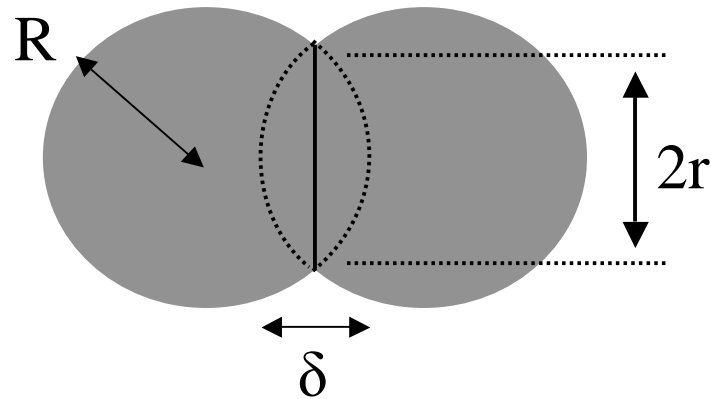
⇒ local restoring force, coupled chains

Groups of W. Kriven, A. Vakakis, J. Lambros



Applications : tunable media ⇒ control of nonlinear stress waves

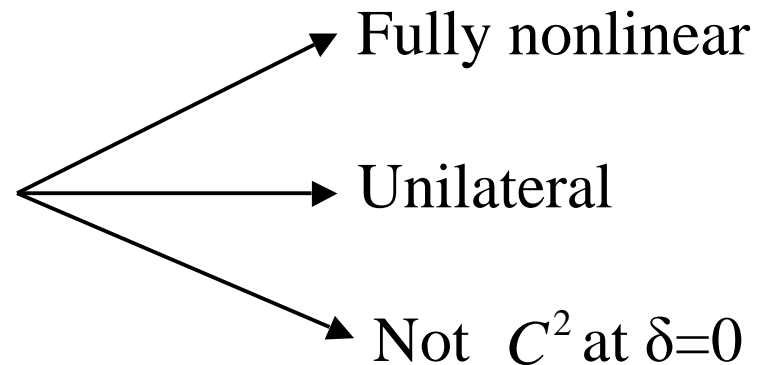
Source of nonlinearity : Hertz contact force



$$\Rightarrow \text{Hertz force} \approx \delta^{3/2}$$

Hooke's law : Hertz force $\approx r \delta$

Geometric nonlinearity : $r \approx \sqrt{R\delta}$

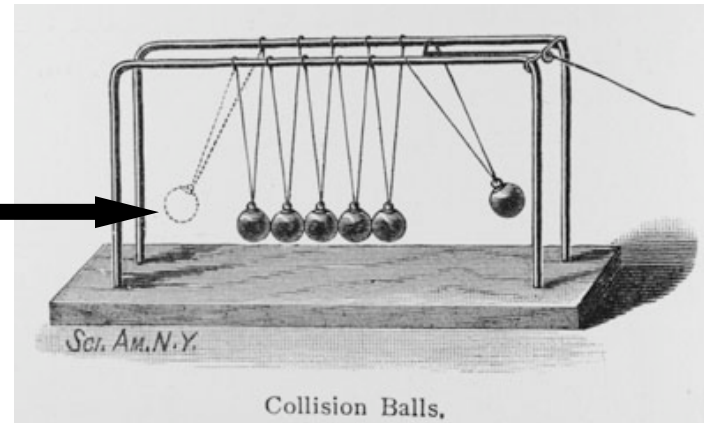


- ⇒ Nonlinear models for chains of beads delicate to analyze: chains of point masses coupled by Hertzian potentials
- ⇒ **Localized waves** can be generated in simulations and experiments: solitary waves, breathers,...

Solitary wave in Newton's cradle

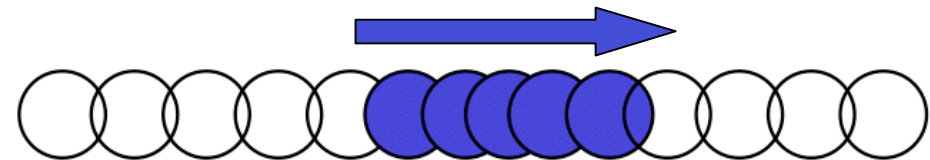
Newton's cradle :

Impacting
bead



bead ejection

Nesterenko (84) :
propagation of a solitary wave



Definition :

$$\text{contact forces} = f(n - ct), \quad \lim_{|\xi| \rightarrow \infty} f(\xi) = 0 \quad (\text{n: bead index})$$

First experimental studies of Nesterenko's solitary wave :

Lazaridi and Nesterenko (85), Coste, Falcon and Fauve (97), Falcon et al (98)

An application : granular shock absorber

Experiments with tapered chains :

impact →

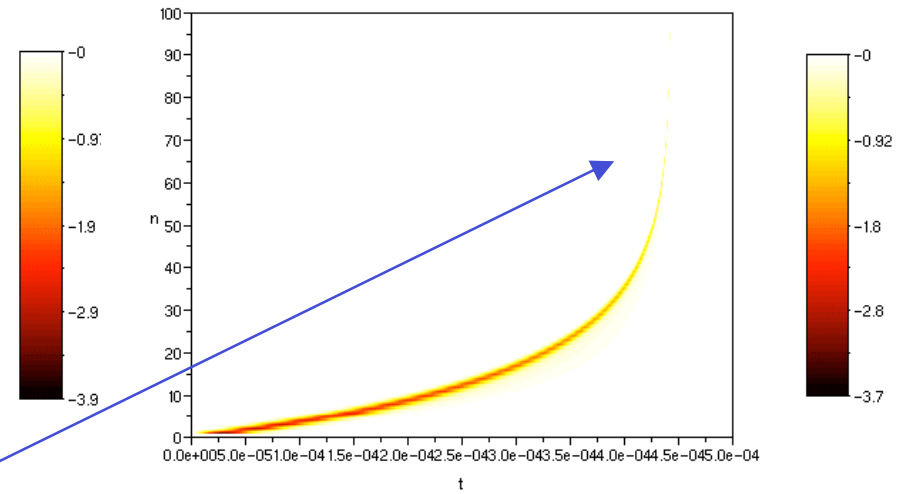
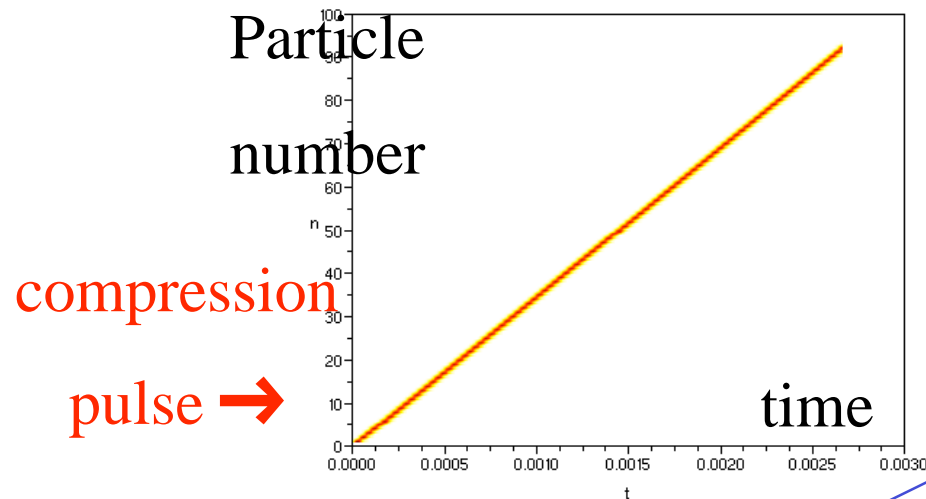


Melo et al,
Phys Rev E 73,
041305 (2006)

Numerical computation of contact forces :

homogeneous chain

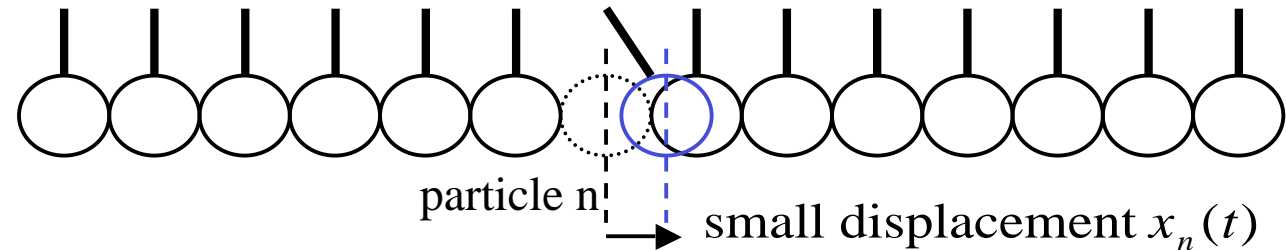
exponentially tapered chain



pulse attenuation during propagation (shock absorber)

Model : Fermi-Pasta-Ulam (FPU) lattice with Hertzian potential

Newton's cradle
under gravity :



$$m \ddot{x}_n + k x_n = \gamma(x_{n-1} - x_n)_+^{3/2} - \gamma(x_n - x_{n+1})_+^{3/2} \quad (a)_+ = \text{Max}(a, 0)$$

Separation of time scales : local oscillations (s), binary collisions (0.1 ms)

⇒ local potential neglected during primary pulse propagation

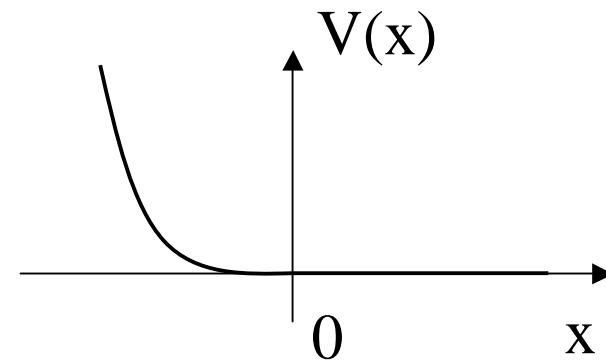
$$m \ddot{x}_n = \gamma(x_{n-1} - x_n)_+^{3/2} - \gamma(x_n - x_{n+1})_+^{3/2} \quad (\text{Fermi-Pasta-Ulam - Hertz})$$

Solitary waves in FPU lattice with Hertzian potential

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \quad n \in \mathbb{Z}$$

$$V(x) = \frac{1}{1+\alpha} (-x)_+^{1+\alpha}$$

$\alpha > 1$ (classical case : $\alpha = 3/2$)



$V''(0) = 0 \Rightarrow$ no sine waves in linearized system ("sonic vacuum")

Existence theorems for solitary waves :

Friesecke-Wattis '94, MacKay '99, Stefanov-Kevrekidis '12

Purely nonlinear Hertz force \Rightarrow unusual properties of solitary waves

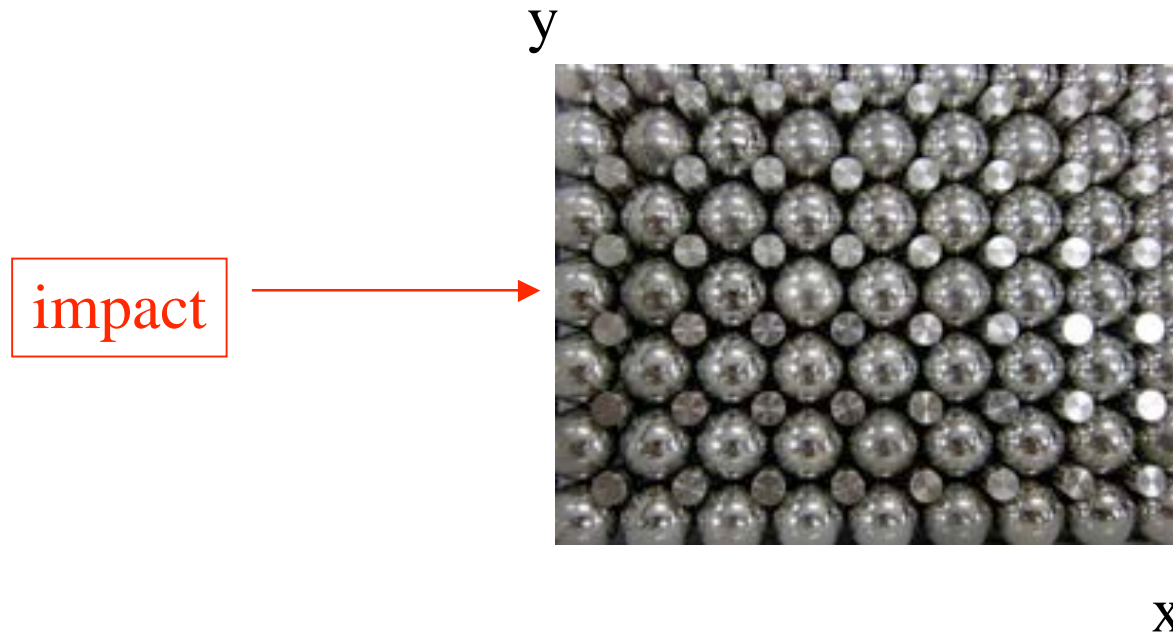
width independent of amplitude, double exponential spatial decay, ...

English-Pego '05, Stefanov-Kevrekidis '12

Shock propagation in higher dimensional granular systems :

C. Daraio's group (Caltech, ETH) :

Experiments on 2D packing of spheres with interstitial cylinders



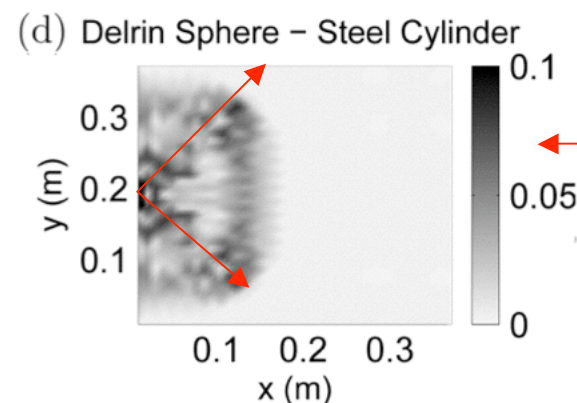
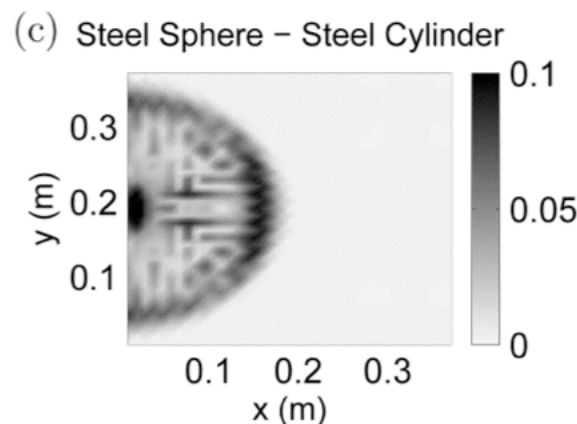
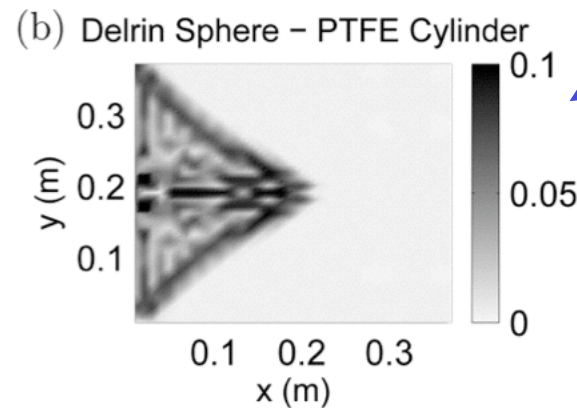
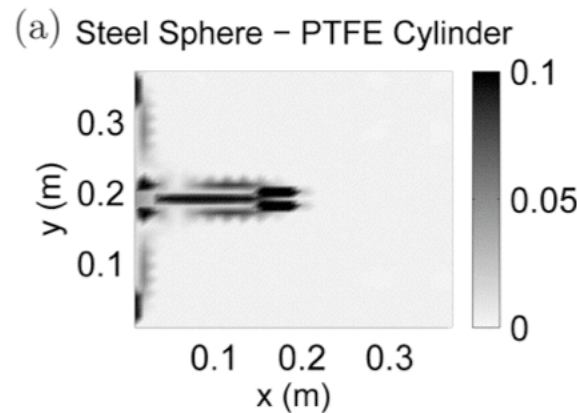
Shock propagation in higher dimensional granular systems :

Different characteristics of stress wave fronts
depending on materials :

Numerical simulations

(particle velocities)

Leonard and Daraio,
PRL 108, 214301 (2012)

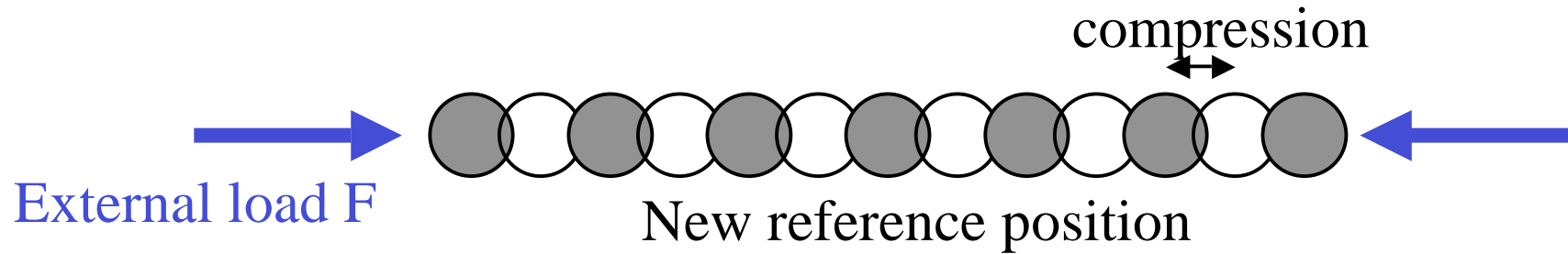


Significant energy
deviation
towards the edges

Breathers in granular chains under precompression

Breathers in diatomic chains - Experiments : Boechler et al '10,

Simulations : Theocharis et al, Phys. Rev. E 82, 2010

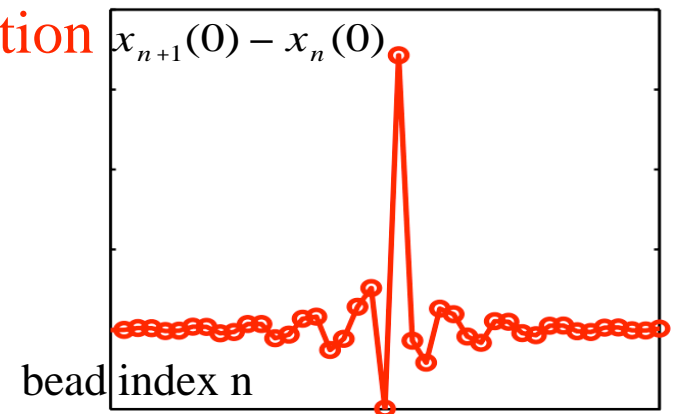


Breather : time-periodic spatially localized oscillation

displacements from reference position :

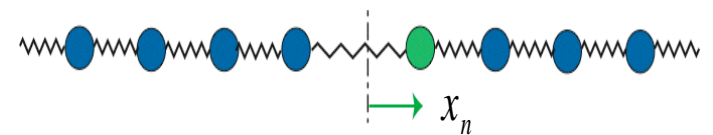
$$x_n(t + T) = x_n(t)$$

$$\lim_{n \rightarrow \pm\infty} x_{n+1}(t) - x_n(t) = 0$$



precompression \Rightarrow classical Fermi - Pasta - Ulam

model : force = $k_F(x_{n+1} - x_n) + O(x_{n+1} - x_n)^2$

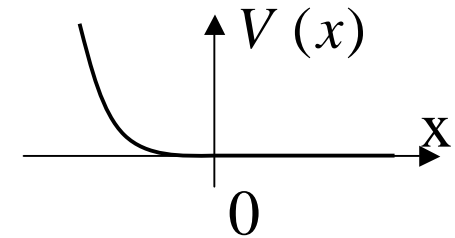


Without precompression : no breathers

$$m_n \ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \quad n \in \mathbb{Z}$$



Hertz potential : **unilateral**



No nontrivial T-periodic breather solutions
satisfying $\lim_{n \rightarrow \pm\infty} \|x_n - x_{n-1}\|_{L^\infty(0,T)} = 0$

J., Kevrekidis, Cuevas,
Physica D 251 (2013)

Quick proof :

average interaction force : $f_n = \frac{1}{T} \int_0^T V'(x_n - x_{n-1}) dt \rightarrow 0$ as $n \rightarrow \infty$ (localization)

f_n independent of $n \Rightarrow f_n = 0$

$V' \leq 0 \Rightarrow V' = 0 \Rightarrow$ beads in free flight $\Rightarrow \frac{dx_n}{dt} = 0 \quad \square$

II - Breathers in Hertzian chains with confining potentials

Outline :

1- Numerical simulations

2- Analysis of energy localization :

asymptotic model for small amplitude solutions :
discrete p -Schrödinger equation

existence of small amplitude breathers

Time-periodic breathers in Hertzian chains with local potentials

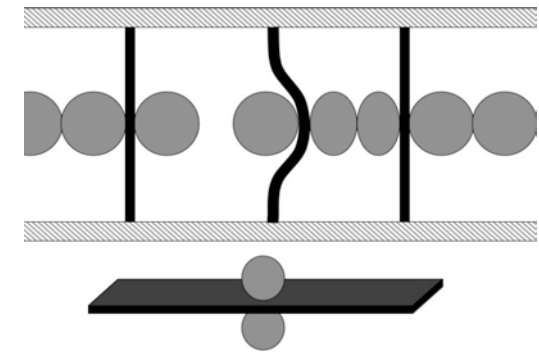
J., Kevrekidis, Cuevas '13

$$\ddot{x}_n + x_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

$$(a)_+ = \text{Max}(a, 0)$$

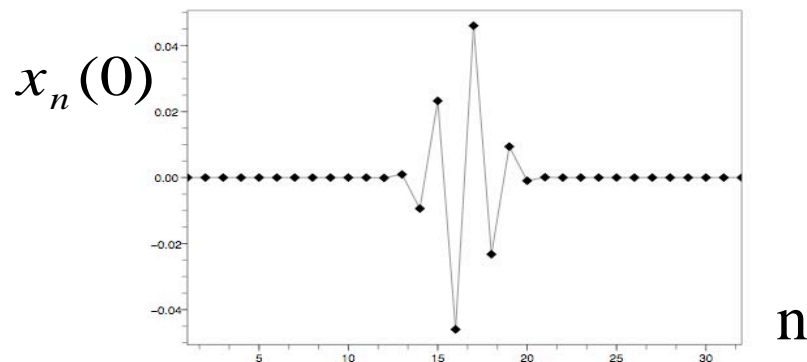
Stiff Newton's cradle

with cantilevers :

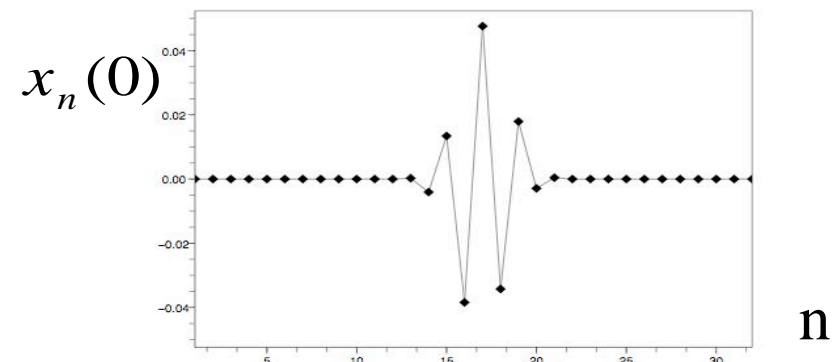


Breather computation by Newton's method :

Bond-centered breather



Site-centered breather

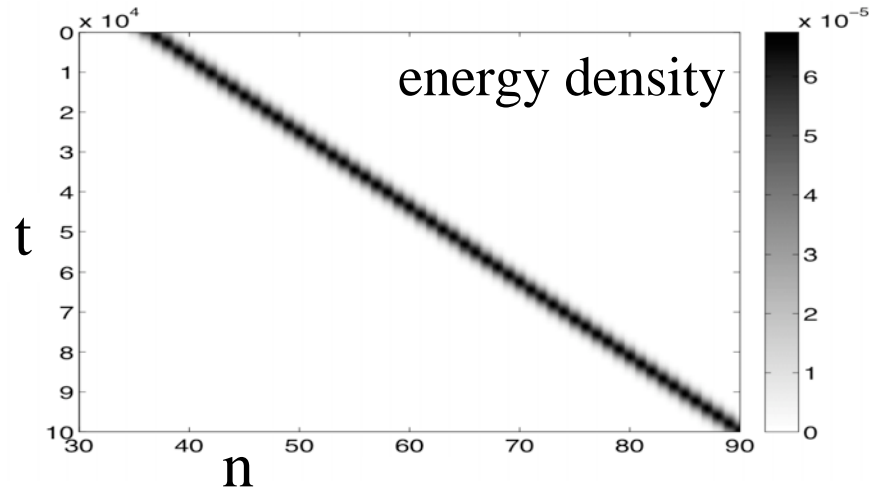


Unusual properties : double exponential decay, high mobility,...

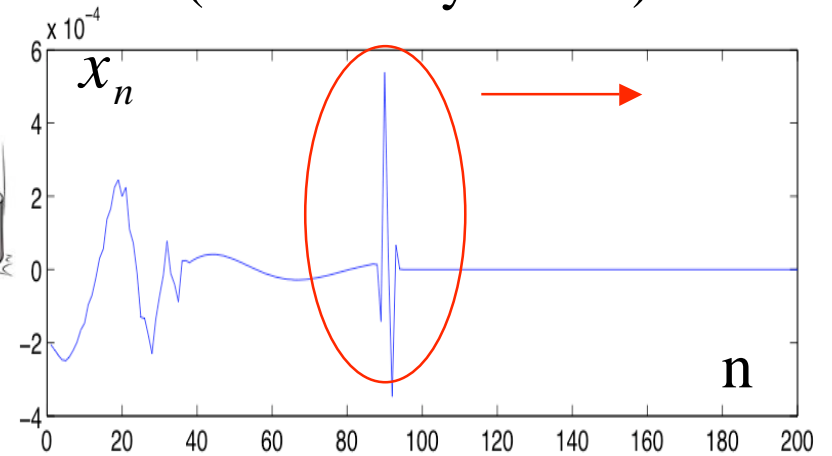
Breather motion

Small perturbation of a stable breather

(energy +0.01%) \Rightarrow traveling breather :



Moderate impacts generate traveling breathers (not solitary waves)

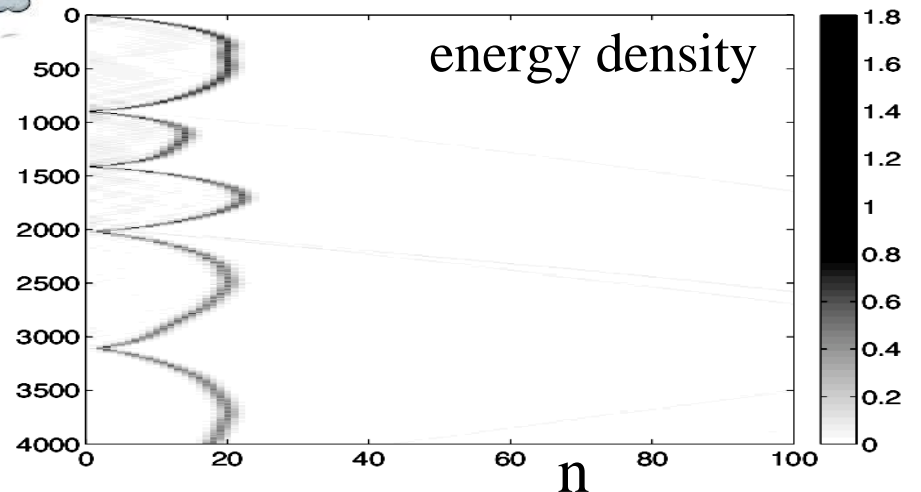


Impact



direction-reversing traveling breather :

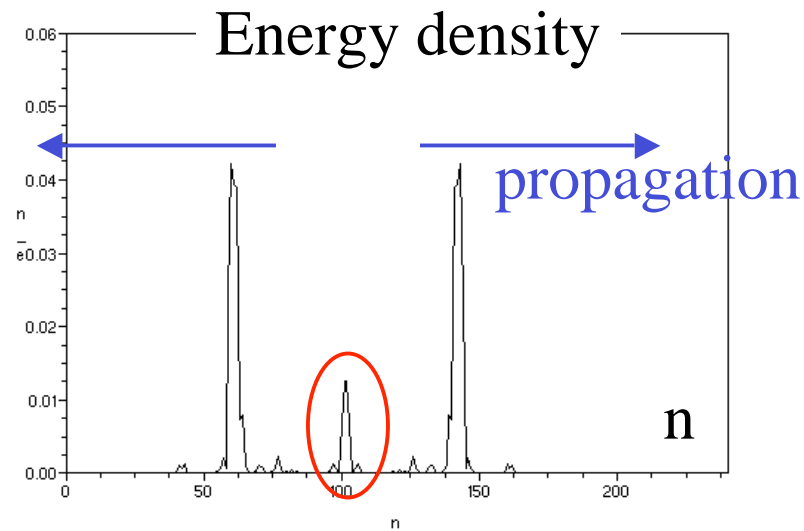
$$\text{hard local potential } \frac{x^2}{2} + \frac{x^4}{4}$$



Static breather generation

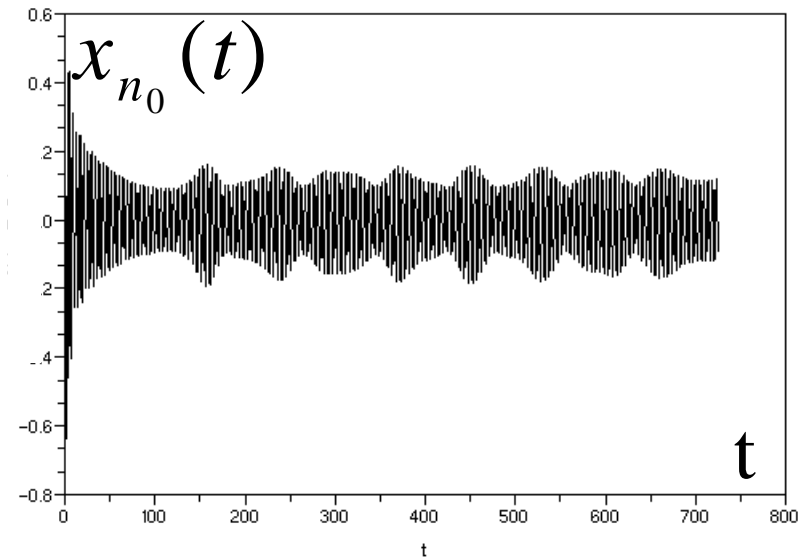
Initial compression of two beads

$$(n = n_0 \text{ and } n = n_0 + 1)$$



Breather
(non-propagating)

Initially compressed bead



Quasiperiodicity ?

Derivation of a simplified asymptotic model :

$$(N.C.) \quad \ddot{x}_n + x_n = (x_{n-1} - x_n)_+^\alpha - (x_n - x_{n+1})_+^\alpha \quad (\alpha > 1)$$

Leading order solutions (small amplitude ε) :

$$x_n^{A,\varepsilon}(t) = \varepsilon A_n(\varepsilon^{\alpha-1} t) e^{it} + \varepsilon \bar{A}_n(\varepsilon^{\alpha-1} t) e^{-it} \quad \text{slow time : } \tau = \varepsilon^{\alpha-1} t$$

Collect terms $O(\varepsilon^\alpha) \times e^{it} \Rightarrow$ **discrete p-Schrödinger (DpS) equation (J.'11)**

$$2\tau_0 i \partial_\tau A_n = (A_{n+1} - A_n) |A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1}) |A_n - A_{n-1}|^{\alpha-1}$$

Continuum limit : $i \partial_\tau A = \partial_\xi (\partial_\xi A | \partial_\xi A |^{p-2})$ with $p = \alpha + 1$ (p -Laplace operator)

- What we \Rightarrow fast oscillations averaged,
- have gained \Rightarrow unilateral character of interactions suppressed by averaging,
- with DpS: \Rightarrow phase invariance, conservation of ℓ_2 norm, scale invariance

Newton's cradle vs discrete p-Schrödinger : error bounds

Infinite chain ($n \in \mathbb{Z}$),

phase space = sequence space ℓ_p with $p \in [1, \infty]$, $\|A\|_p = \left(\sum_{n=-\infty}^{+\infty} |A_n|^p \right)^{1/p}$

The DpS equation approximates true $O(\varepsilon)$ solutions of N.C.

up to an error $O(\varepsilon^\alpha)$, over long times $O(\varepsilon^{1-\alpha})$:

Theorem (Bidégaray-Fesquet, Dumas, J. '13)

Fix a solution of DpS : $A_n(\tau) : [0, T] \rightarrow \ell_p(\mathbb{Z})$

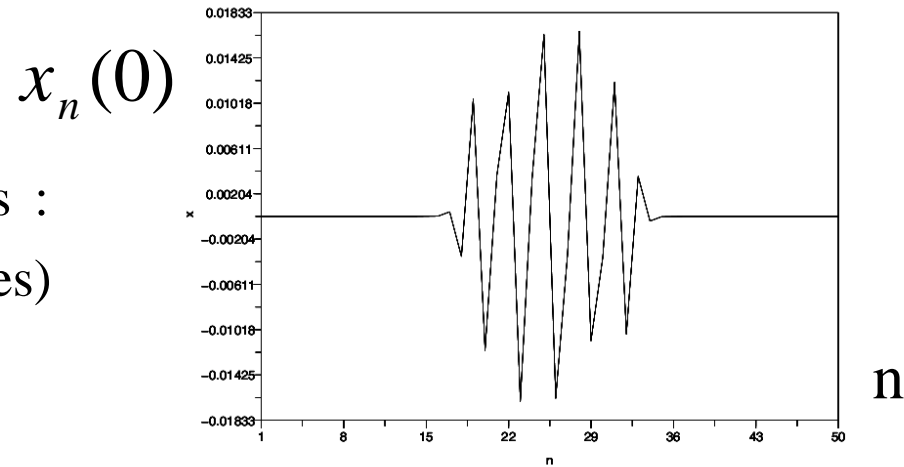
For all ε small enough, the solutions of N.C. with initial conditions

$(x_n(0), \dot{x}_n(0))_n = (x_n^{A, \varepsilon}(0), \dot{x}_n^{A, \varepsilon}(0))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ satisfy :

$(x_n(t), \dot{x}_n(t))_n = (x_n^{A, \varepsilon}(t), \dot{x}_n^{A, \varepsilon}(t))_n + O(\varepsilon^\alpha)$ in $\ell_p^2(\mathbb{Z})$ uniformly in $t \in [0, T\varepsilon^{1-\alpha}]$

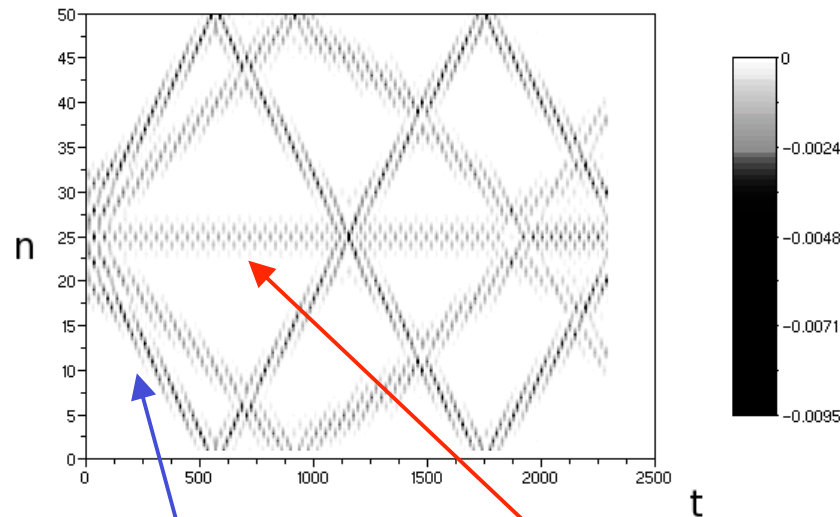
Numerical comparison N.C. - DpS ($\alpha=3/2$)

Initial displacements :
(zero initial velocities)



Interaction forces (grey levels) :

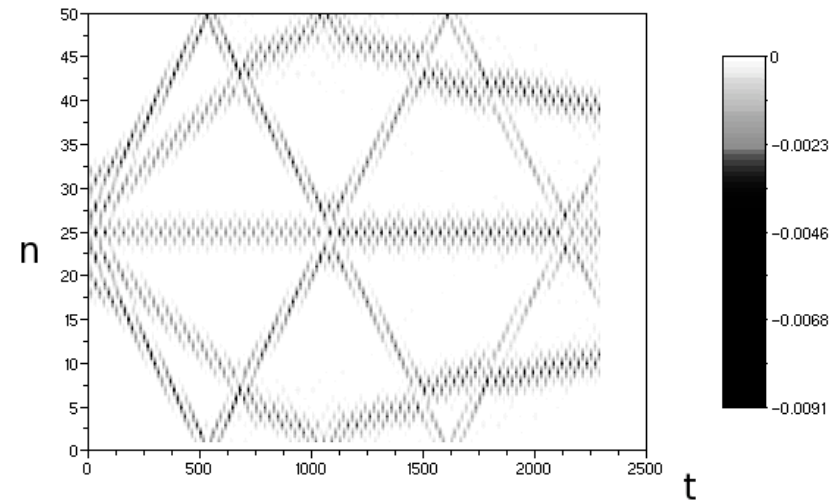
Newton's cradle :



traveling breather

breather (non-propagating)

DpS approximation:



Localized solutions of the DpS equation :

1 - Absence of scattering

DpS equation ($\alpha > 1$) :

$$i \partial_{\tau} A_n = (A_{n+1} - A_n) |A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1}) |A_n - A_{n-1}|^{\alpha-1}$$

Theorem (Bidégaray-Fesquet, Dumas, J. '13) :

DpS is globally well-posed in $\ell_2(\mathbb{Z})$. If $A(0) \neq 0$, then

$$\text{for all times : } \|A(\tau)\|_{\infty} \geq \left(\frac{\left\| \frac{1}{2} \delta^+ A(0) \right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2} \right)^{\frac{1}{\alpha-1}}$$

$$(\delta^+ A)_n = A_{n+1} - A_n, \quad \|A\|_p = \left(\sum_{n=-\infty}^{+\infty} |A_n|^p \right)^{1/p}, \quad \|A\|_{\infty} = \sup_n |A_n|$$

Proof of absence of scattering for DpS equation :

Use conservation of energy $H = \sum_{n=-\infty}^{+\infty} |A_{n+1} - A_n|^{\alpha+1}$ and $\|A\|_2^2 = \sum_{n=-\infty}^{+\infty} |A_n|^2$

$$\begin{aligned} \|\delta^+ A(0)\|_{\alpha+1} &= H^{\frac{1}{\alpha+1}} = \|\delta^+ A(\tau)\|_{\alpha+1} && \text{(energy conservation)} \\ &\leq 2\|A(\tau)\|_{\alpha+1} && \text{(triangular inequality)} \\ &\leq 2\left(\|A(\tau)\|_{\infty}\right)^{1-\frac{2}{1+\alpha}} \left(\|A(\tau)\|_2\right)^{\frac{2}{1+\alpha}} && \text{(interpolation inequality)} \\ &\leq 2\left(\|A(\tau)\|_{\infty}\right)^{1-\frac{2}{1+\alpha}} \left(\|A(0)\|_2\right)^{\frac{2}{1+\alpha}} && \text{(conserved } \ell_2 \text{ norm)} \end{aligned}$$

$$\Rightarrow \|A(\tau)\|_{\infty} \geq \left(\frac{1}{2}\|\delta^+ A(0)\|_{\alpha+1}\right)^{\frac{\alpha+1}{\alpha-1}} \left(\|A(0)\|_2\right)^{\frac{2}{1-\alpha}} = \left(\frac{\left\|\frac{1}{2}\delta^+ A(0)\right\|_{\alpha+1}^{\alpha+1}}{\|A(0)\|_2^2}\right)^{\frac{1}{\alpha-1}}$$

□

Localized solutions of the DpS equation : 2 - Breather solutions (time-periodic)

$$\text{DpS} : i \partial_{\tau} A_n = (A_{n+1} - A_n) |A_{n+1} - A_n|^{\alpha-1} - (A_n - A_{n-1}) |A_n - A_{n-1}|^{\alpha-1}$$

Time-periodic solutions to DpS : $A_n(\tau) = a_n e^{i\tau}$ $a_n \in \mathbb{R}$

Stationary (real) DpS equation :

$$-a_n = (a_{n+1} - a_n) |a_{n+1} - a_n|^{\alpha-1} - (a_n - a_{n-1}) |a_n - a_{n-1}|^{\alpha-1}$$

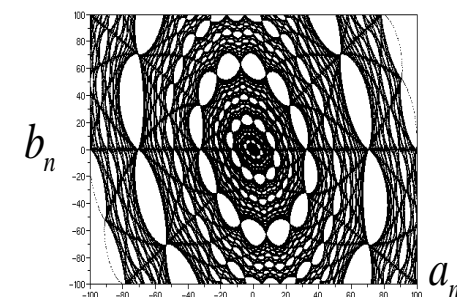
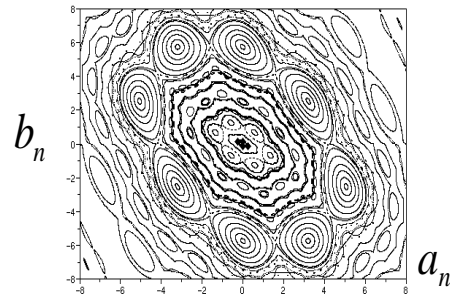
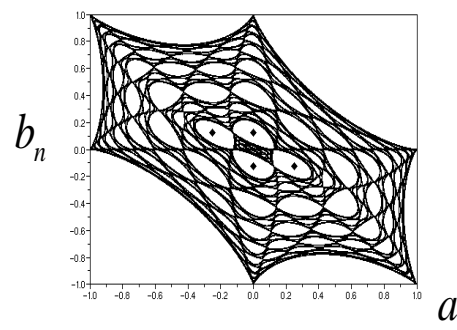
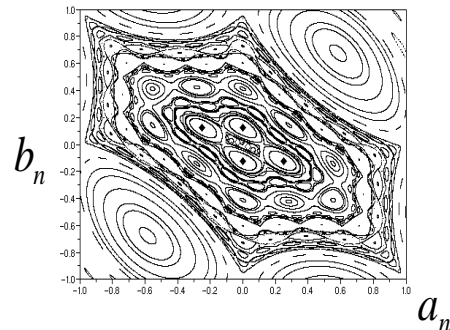
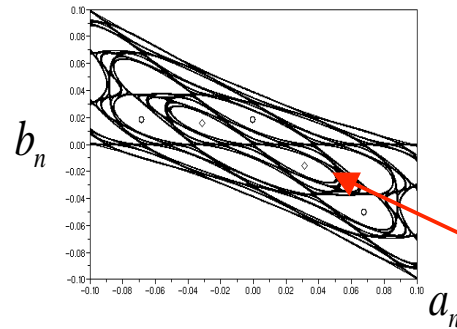
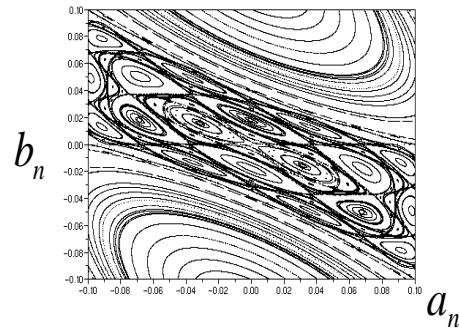
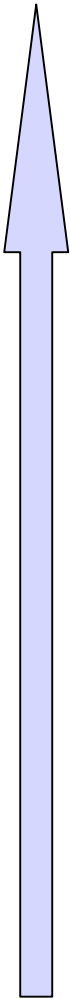
b_n

Spatial dynamics : stationary real DpS \Leftrightarrow 2D mapping

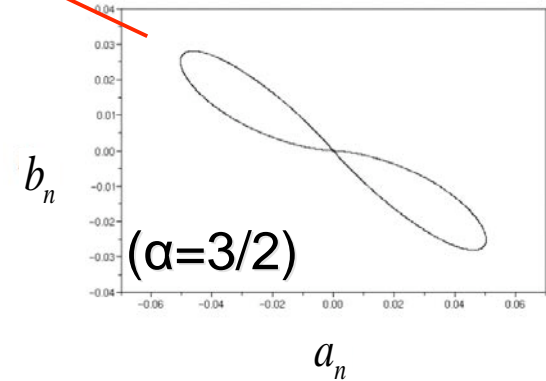
$$(a_{n+1}, b_{n+1}) = G(a_n, b_n) \quad \text{G reversible, area - preserving, not differentiable at the origin}$$

Stationary Dps equation : some orbits of the « spatial map » G

Zoom
towards
(0,0)



Stable and unstable
manifolds of (0,0)
intersect :

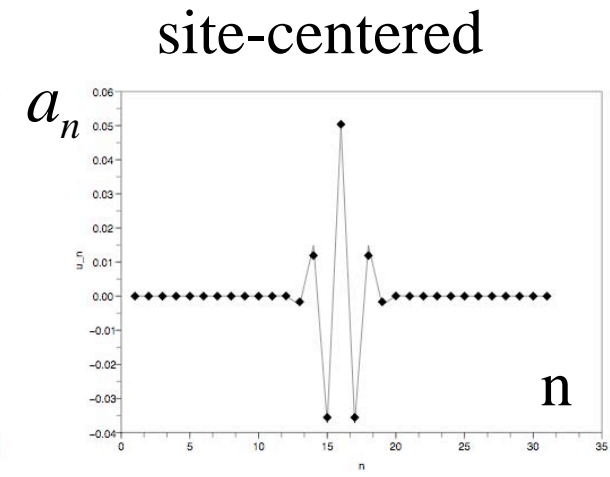
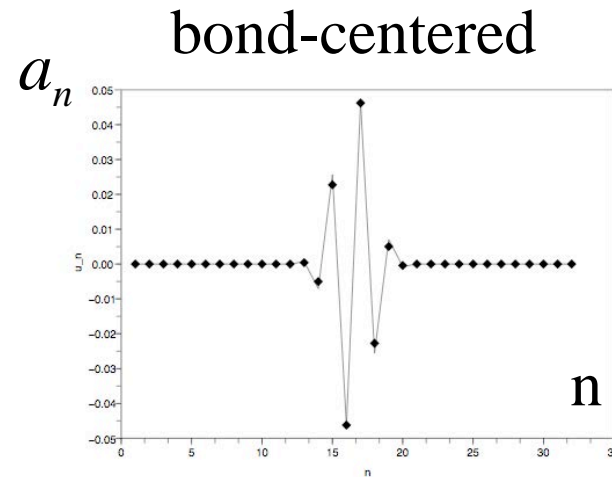


\Rightarrow orbits
homoclinic to 0

$$(a_n, b_n) \longrightarrow 0$$

$$n \rightarrow \pm\infty$$

Theorem :
 existence of reversible
 homoclinics
 (J. and Starosvetsky '12)



Method : $y_n = (-1)^n b_n$ solution of generalized stationary discrete NLS equation

$$y_{n+1} - 2y_n + y_{n-1} + U'(y_n) = 0 \Rightarrow \text{shooting method (Qin and Xiao '07)}$$

$$U(y) = -\frac{\alpha}{\alpha+1} |y|^{1+1/\alpha} + 2y^2 \text{ nonsmooth double - well potential}$$

Orbits homoclinic to 0
 for stationary DpS equation

\Rightarrow long-lived breather
 solutions of N.C. :

$$x_n(t) = 2\varepsilon a_n \cos\left[\left(1 + \frac{\varepsilon^{\alpha-1}}{2\tau_0}\right)t\right] + O(\varepsilon^\alpha)$$

over long times $t \approx \varepsilon^{1-\alpha}$

Conclusion

We have studied granular chains with local potentials
(beads attached to stiff cantilevers, cables, elastic matrix...)

- ❖ model : nonlinear lattice, Hertzian interactions and local potential
- ❖ strongly localized breathers (static/moving) easily generated in simulations
- ❖ new asymptotic model : **discrete p -Schrödinger equation**
 - in Newton's cradle : describes small solutions over long times
 - absence of scattering, existence of time-periodic breathers

Future directions :

incorporate dissipation and spatial inhomogeneities,

experimental study of breathers and applications

THANK YOU !

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J., Kevrekidis, Cuevas, *Physica D* 251 (2013), 39-59

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Asymptotic model (p-Schrödinger), error bounds, breathers :

J., *Math. Models Meth. Appl. Sci.* 21 (2011), 2335-2377

J., Starosvetsky (2012), to appear in *Nonlinear Systems and Complexity (Springer)*, arXiv:1307.8324

Bidégaray-Fesquet, Dumas, J. (2013), to appear in *SIAM J. Math. Anal.*, arXiv:1306.2105

Further reading

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Starosvetsky et al, SIAM J. Appl. Math. 72 (2012), 337-361

P. LaVigne, MSc Thesis, Mechanical Engineering, University of Illinois at Urbana-Champaign, 2012, <http://hdl.handle.net/2142/34243>

Hasan et al, Int. J. Solids Str. 50 (2013), 3207-3224

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Herrmann, Proc. R. Soc. Edinb. Sect. A-Math. 140 (2010), 753-785

Stefanov and Kevrekidis, J. Nonlinear Sci. 22 (2012), 327-349

J. and Pelinovsky, Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, 2013, arXiv:1307.3837

Further reading

Reviews and monographs on solitary waves and impacts
in granular chains :

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Lecture Notes in Applied and Computational Mechanics 72, Springer, 2014

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E. Falcon, Comportements dynamiques associés au contact de Hertz :
processus collectifs de collision et propagation d'ondes solitaires dans les
milieux granulaires, PhD thesis, Université Claude Bernard Lyon 1 (1997)