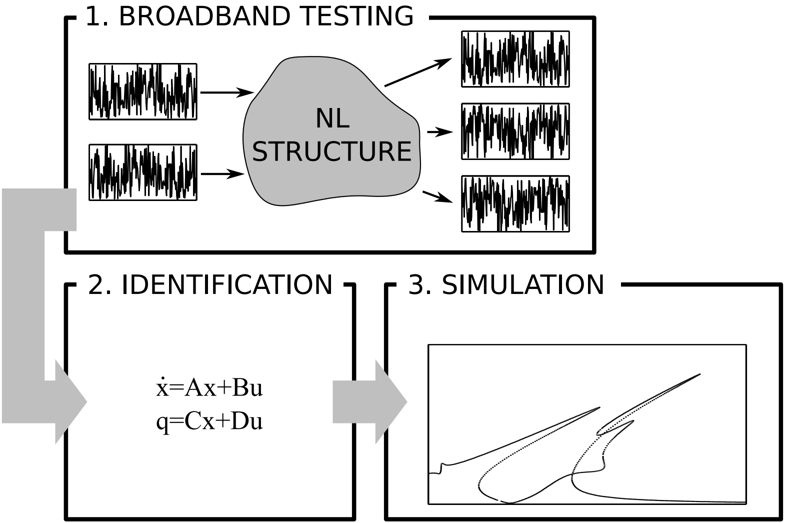
Nonlinear frequency response curve from broadband testing

Etienne Gourc, Chiara Grappasonni,
Jean-Philippe Noel, Gaetan Kerschen

Space Structures and Systems Lab (S3L)
Department of Aerospace and Mechanical Engineering
University of Liège, Belgium

Three step procedure



Feedback interpretation of nonlinearities

General nonlinear system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) + f(q(t), \dot{q}(t)) = p(t)$$

Feedback interpretation of nonlinearities

General nonlinear system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) + f(q(t), \dot{q}(t)) = p(t)$$

Expression of the nonlinear terms

$$f(q(t), \dot{q}(t)) = \sum_{j=1}^{s} \mu_j L_j g_j(q(t), \dot{q}(t))$$

Feedback interpretation of nonlinearities

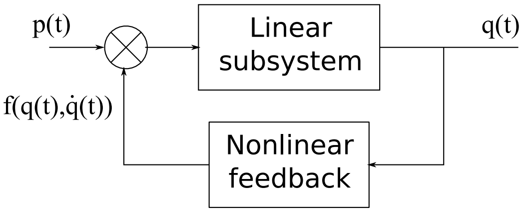
General nonlinear system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) + f(q(t), \dot{q}(t)) = p(t)$$

Expression of the nonlinear terms

$$f(q(t), \dot{q}(t)) = \sum_{j=1}^{s} \mu_j L_j g_j(q(t), \dot{q}(t))$$

Feedback interpretation



From physical to state-space domain

State and extended input vectors

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \quad e(p(t), q(t), \dot{q}(t)) = \begin{bmatrix} p(t) \\ g_1(q(t), \dot{q}(t)) \\ \vdots \\ g_s(q(t), \dot{q}(t)) \end{bmatrix}$$

From physical to state-space domain

State and extended input vectors

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \quad e(p(t), q(t), \dot{q}(t)) = \begin{bmatrix} p(t) \\ g_1(q(t), \dot{q}(t)) \\ \vdots \\ g_s(q(t), \dot{q}(t)) \end{bmatrix}$$

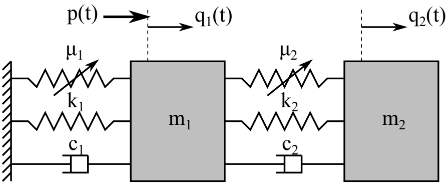
Nonlinear state space system

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c e(p(t), q(t), \dot{q}(t)) \\ q(t) &= C x(t) + D e(p(t), q(t), \dot{q}(t)) \end{aligned}$$

$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 & 0 & \dots & 0 \\ M^{-1}L_f & -\mu_1 M^{-1} L_1 & \dots & -\mu_s M^{-1} L_s \end{bmatrix}, \quad D = 0$$

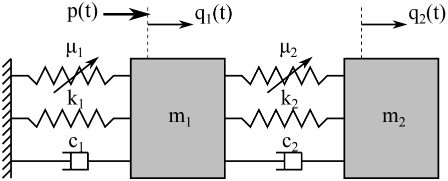
Identification statement



$$e(t) = \begin{bmatrix} p(t) \\ q_1(t)^3 \\ (q_1(t) - q_2(t))^3 \end{bmatrix},\ L_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ L_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Identification statement



$$e(t) = \begin{bmatrix} p(t) \\ q_1(t)^3 \\ (q_1(t) - q_2(t))^3 \end{bmatrix}, \; L_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Given

- Measured output $q(t)$
- Measured input $p(t)$
- Set of basis fuction $g_j(q(t), \dot{q}(t))$

Determine

- Order of the system n
- System matrices A_c, B_c, C, D
- Nonlinear coefficients μ_j

Subspace identification method

Discrete time translation

$$
\begin{aligned}
x(k+1) &= A_d x(k) + B_d e(k) \\
q(k) &= C x(k) + D e(k)
\end{aligned}
$$

Subspace identification method

Discrete time translation

$$
\begin{aligned}
x(k+1) &= A_d x(k) + B_d e(k) \\
q(k) &= Cx(k) + De(k)
\end{aligned}
$$

Block Hankel matrices

$$
Q_i = \underbrace{}_{}\overset{\xleftrightarrow{\quad j \quad}}{ri\updownarrow\begin{bmatrix} q(1) & q(2) & \dots & q(j) \\ q(2) & q(3) & \dots & q(j+1) \\ \vdots & \vdots & \ddots & \vdots \\ q(i) & q(i+1) & \dots & q(i+j-1) \end{bmatrix}}
$$

$$
E_i = \begin{bmatrix} e(1) & e(2) & \dots & e(j) \\ e(2) & e(3) & \dots & e(j+1) \\ \vdots & \vdots & \ddots & \vdots \\ e(i) & e(i+1) & \dots & e(i+j-1) \end{bmatrix}
$$

Input-State-Output equation

$$Q_i = \Gamma_i X + H_i E_i$$

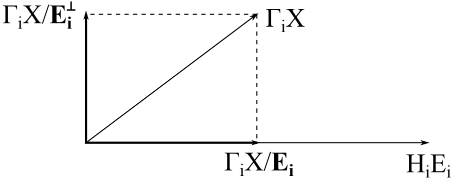
with

$$\Gamma_i = \begin{bmatrix} C^T & (CA_d)^T & (CA_d^2)^T & \dots & (CA_d^{i-1})^T \end{bmatrix}^T$$

$$H_i = \begin{bmatrix} D & 0 & 0 & \dots & 0 \\ CB_d & D & 0 & \dots & 0 \\ CA_dB_d & CB_d & D & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA_d^{i-2}B_d & CA_d^{i-3}B_d & CA_d^{i-4}B_d & \dots & D \end{bmatrix}$$

Projection algorithm

Orthogonal projection

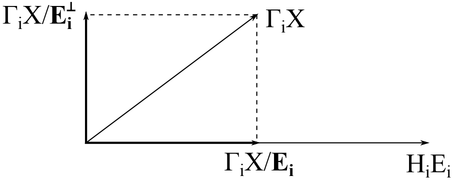


$$Q_i = \Gamma_i X + H_i E_i$$

$$Q_i/\mathbf{E_i}^\perp = \Gamma_i X/\mathbf{E_i}^\perp = \mathcal{O}_i$$

Projection algorithm

Orthogonal projection



$$Q_i = \Gamma_i X + H_i E_i$$

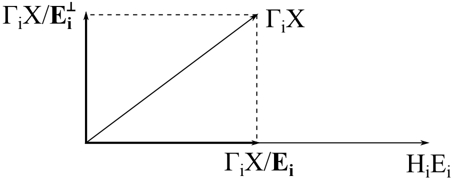
$$Q_i / \mathbf{E_i}^{\perp} = \Gamma_i X / \mathbf{E_i}^{\perp} = \mathcal{O}_i$$

Singular value decomposition

$$\mathcal{O}_i = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

Projection algorithm

Orthogonal projection



$$Q_i = \Gamma_i X + H_i E_i$$

$$Q_i/\mathbf{E_i}^{\perp} = \Gamma_i X/\mathbf{E_i}^{\perp} = \mathcal{O}_i$$

Singular value decomposition

$$\mathcal{O}_i = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$$\begin{cases} \Gamma_i = U_1 S_1^{1/2} T \\ X/\mathbf{E_i}^{\perp} = T^{-1} S_1^{1/2} V_1^T \end{cases}$$

Extraction of system matrices

A_d and C

$$\Gamma_i = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A}_d \\ \hat{C}\hat{A}_d^2 \\ \vdots \\ \hat{C}\hat{A}_d^{i-1} \end{bmatrix}, \ \underline{\Gamma_i} = \hat{A}_d \overline{\Gamma_i}$$

B_d and D

$$Q_i = \Gamma_i X + H_i E_i$$

$$\Gamma_i^{\perp} Q_i = \Gamma_i^{\perp} H_i E_i$$

$$A_d = T\hat{A}_d T^{-1}, \quad B_d = T\hat{B}_d, \quad C = \hat{C}T^{-1}, \quad D = \hat{D}$$

Extraction of system matrices

A_d and C

$$\Gamma_i = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A}_d \\ \hat{C}\hat{A}_d^2 \\ \vdots \\ \hat{C}\hat{A}_d^{i-1} \end{bmatrix}, \ \underline{\Gamma_i} = \hat{A}_d \overline{\Gamma_i}$$

B_d and D

$$Q_i = \Gamma_i X + H_i E_i$$

$$\Gamma_i^{\perp} Q_i = \Gamma_i^{\perp} H_i E_i$$

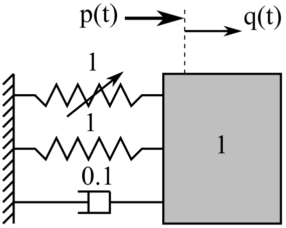
$$A_d = T\hat{A}_d T^{-1}, \quad B_d = T\hat{B}_d, \quad C = \hat{C}T^{-1}, \quad D = \hat{D}$$

Nonlinear coefficients (Optional)

$$H_E(\omega) = \underbrace{\hat{D} + \hat{C}(j\omega I - \hat{A}_c)^{-1}\hat{B}_c}_{\text{Invariant with } T} = \begin{bmatrix} H & -H\mu_1 L_1 & \dots & -H\mu_s L_s \end{bmatrix}$$

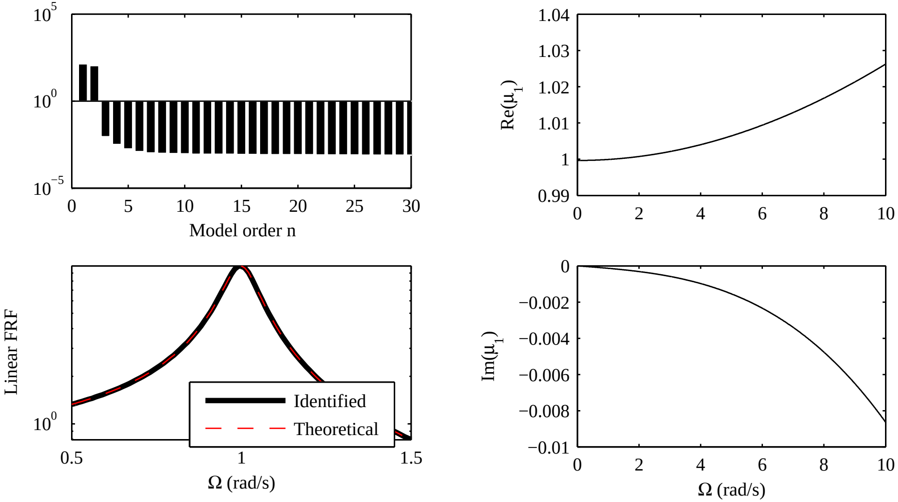
H : FrF of underlying linear system

Application : Duffing oscillator



$$\overbrace{e(t) = \begin{bmatrix} p(t) \\ q(t)^3 \end{bmatrix}}^{\text{Input}} \rightarrow \boxed{\begin{matrix} n \\ A_c \; B_c \\ C \; D \\ \mu \end{matrix}} \rightarrow \overbrace{q(t)}^{\text{Output}}$$

Result of identification



Harmonic balance method

Continuation in state space domain

$$
\begin{aligned}
\dot{x}(t) &= A_c x(t) + B_c e(p(t), q(t), \dot{q}(t)) \\
q(t) &= C x(t) + D e(p(t), q(t), \dot{q}(t))
\end{aligned}
$$

Harmonic balance method

Continuation in state space domain

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + B_c e(p(t), q(t), \dot{q}(t)) \\ q(t) &= C x(t) + D e(p(t), q(t), \dot{q}(t))\end{aligned}$$

Truncated Fourier series expansion

$$x(t) = \frac{X_0}{\sqrt{2}} + \sum_{j=1}^{N} X_{cj} \cos(k_j \Omega t) + X_{sj} \sin(k_j \Omega t)$$

$$q(t) = \frac{Q_0}{\sqrt{2}} + \sum_{j=1}^{N} Q_{cj} \cos(k_j \Omega t) + Q_{sj} \sin(k_j \Omega t)$$ Unknown Fourier coeffs Q

$$e(t) = \frac{E_0}{\sqrt{2}} + \sum_{j=1}^{N} E_{cj} \cos(k_j \Omega t) + E_{sj} \sin(k_j \Omega t)$$ Fourier coeffs E Depend on Q

Harmonic balance method

Residue equation

$$h(Q,\Omega) \equiv \underbrace{Q - G(\Omega)E(Q)}_{\text{Invariant with } T} = 0$$

with

$$G(\Omega) = (I \otimes C)\Lambda^{-1}(I \otimes B_c) + (I \otimes D)$$

$$\Lambda = \Omega(\nabla \otimes I) - (I \otimes A_c)$$

Harmonic balance method

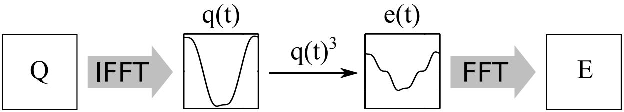
Residue equation

$$h(Q,\Omega) \equiv \underbrace{Q - G(\Omega)E(Q)}_{\text{Invariant with } T} = 0$$

with

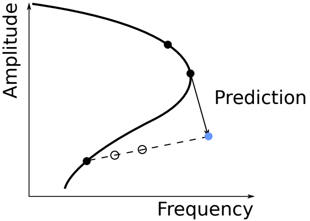
$$G(\Omega) = (I \otimes C)\Lambda^{-1}(I \otimes B_c) + (I \otimes D)$$
$$\Lambda = \Omega(\nabla \otimes I) - (I \otimes A_c)$$

Fourier coefficients of nonlinear terms



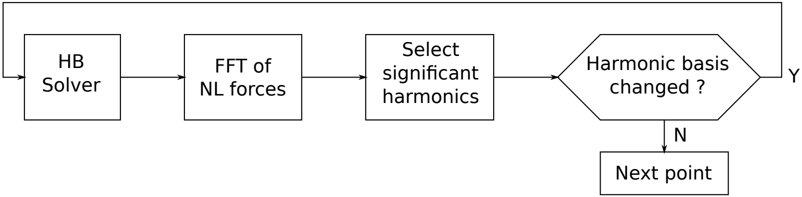
Tracking periodic solutions

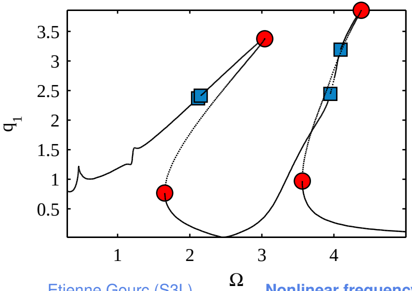
Pseudo-arclength continuation

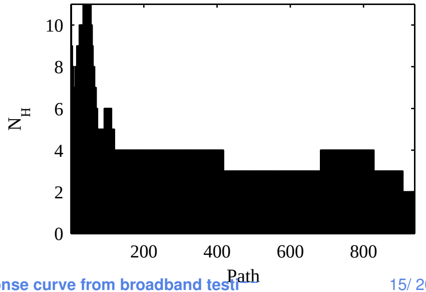


Robust implementation

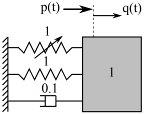
Automatic harmonic selection

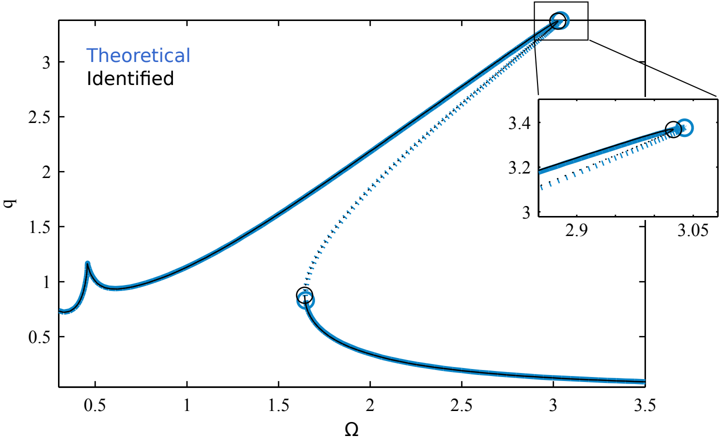




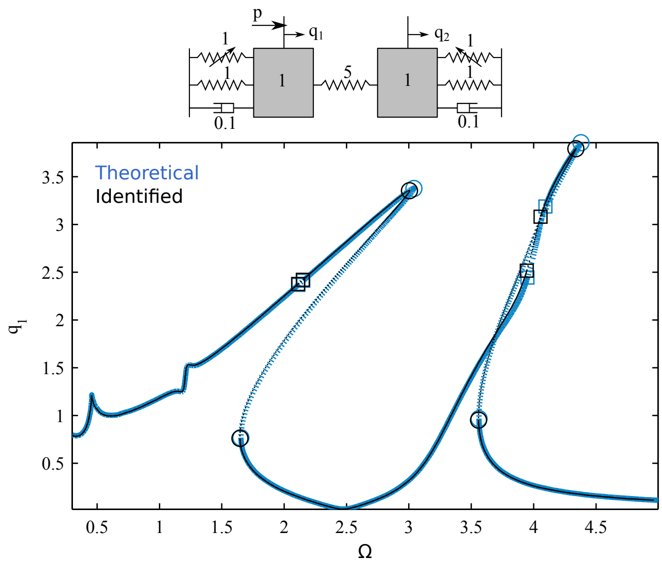


Application 1 : single dof Duffing





Application 2 : two dof Duffing



Conclusion

- Nonlinear subspace identification + Continuation
- Good results on simple systems

Conclusion

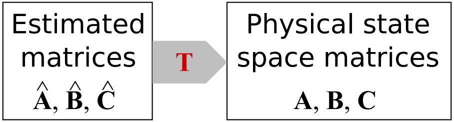
- Nonlinear subspace identification + Continuation
- Good results on simple systems

Perspectives

- Effect of noise on the detection of bifurcation
- Experimental validation

Error identification

Going back to physics



$$T \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} T$$

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} T$$

This gives

$$T = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix}$$

Identified matrices in ”physical space”

$$A_d = T\hat{A}_d T^{-1}, \quad B_d = T\hat{B}_d, \quad C = \hat{C}T^{-1}, \quad D = \hat{D}$$

Uncorrect B_c matrix

$$B_c = \begin{bmatrix} B_1 & B_2 \\ M^{-1}L_f & -M^{-1}L_1 \end{bmatrix}$$

This gives for the Duffing

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q + \frac{\mu}{m}q^3 - B_2\dot{q}q^2 = \frac{p}{m} + B_1\dot{p}$$